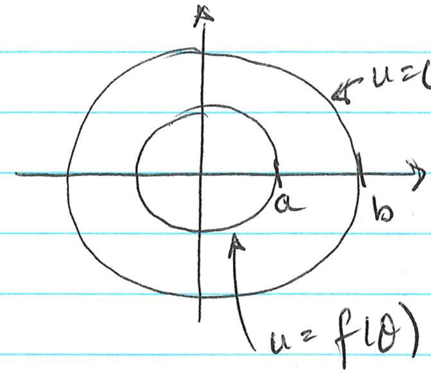
Ex 2:

$$\begin{cases} \nabla^2 u = 0 & a < r < b,\ -\pi < \theta < \pi \\ u(a,\theta) = f(\theta) & \\ u(b,\theta) = 0 & \end{cases} \Big\} -\pi < \theta < \pi$$



$*$ periodic domain!!

$$\boxed{\therefore\ u(-\pi) = u(\pi) \qquad \frac{\partial u}{\partial \theta}(-\pi) = \frac{\partial u}{\partial \theta}(\pi)}$$

Sol'n: step 1 Separate (see example 1)

step 2: ODEs

Ⓐ $\begin{cases} T'' + \lambda T = 0 \\ T(-\pi) = T(\pi) \\ T'(-\pi) = T'(\pi) \end{cases}$ Ⓑ $r^2 R'' + rR' - \lambda R = 0$

step 3: solve Ⓐ + Ⓑ

Ⓐ Non-trivial solutions are obtained for $\lambda = 0$ and $\lambda > 0$ $(\lambda = \omega^2)$.

Case i: $\lambda = 0$

$T(\theta) = c_1\theta + c_2$

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BCs:

$$\begin{cases} T(-\pi) = T(\pi) \\ T'(-\pi) = T'(\pi) \end{cases} \Leftrightarrow \begin{cases} c_1(-\pi) + c_2 = c_1(\pi) + c_2 \\ c_1 = c_1 \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 \text{ arbitrary} \end{cases}$$

$\therefore\ T(\theta) = K$, a constant

Case ii: $\lambda = \omega^2 > 0$

$$T(\theta) = c_1 \cos(\omega\theta) + c_2 \sin(\omega\theta)$$
$$T'(\theta) = -\omega c_1 \sin(\omega\theta) + \omega c_2 \cos(\omega\theta)$$

BCs:

$$\begin{cases} T(-\pi) = T(\pi) \\ T'(-\pi) = T'(\pi) \end{cases} \Leftrightarrow \begin{cases} c_1 \cancel{\cos(\omega\pi)} - c_2 \sin(\omega\pi) = c_1 \cancel{\cos(\omega\pi)} + c_2 \sin(\omega\pi) \\ \omega c_1 \sin(\omega\pi) + \omega c_2 \cos(\omega\pi) \\ \quad = -\omega c_1 \sin(\omega\pi) + \cancel{\omega c_2 \cos(\omega\pi)} \end{cases}$$

$$\Leftrightarrow \begin{cases} 2 c_2 \sin(\omega\pi) = 0 \\ 2 \omega c_1 \sin(\omega\pi) = 0 \end{cases}$$

$\therefore$ for nontrivial sol'ns we require

$$\omega\pi = n\pi \Leftrightarrow \omega = n, \quad n \in \mathbb{N}$$

and the eigenfunctions are

$$T_n(\theta) = c_{1n} \cos(n\theta) + c_{2n} \sin(n\theta)$$

(B) Case i: $\lambda = 0$

$$r^2 R'' + r R' = 0$$

Let $V = R'$. Then we have

$$r^2 V' + r V = 0 \Leftrightarrow \frac{V'}{V} = -\frac{1}{r}$$

$$\Leftrightarrow \int \frac{dV}{V} = \int -\frac{1}{r}\, dr$$

$$\Leftrightarrow \ln|V| = -\ln|r| + \bar{K} = \ln\left|\frac{1}{r}\right| + \bar{K}$$

$$\Leftrightarrow V = \frac{\tilde{K}}{r}$$

But $V = R'$ and so

$$R' = \frac{\tilde{K}}{r} \Leftrightarrow R = \tilde{K}\ln(r) + \hat{K}$$

where $\tilde{K}$ and $\hat{K}$ are arbitrary constants. We have one homogeneous BC: $R(b) = 0 \Leftrightarrow \hat{K} = -\tilde{K}\ln(b)$

$$\therefore R(r) = \tilde{K}\ln(r/b)$$

Case ii: $\lambda = \omega^2 > 0$

$$R_n(r) = d_{1n} r^n + d_{2n} r^{-n} \quad \text{(see Ex 1)}$$

$\because$ the domain does not have points $r \to 0$, both d_{1n} & d_{2n} can be included.

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Step 4: Superposition

$$u(r,\theta) = T_0 R_0 + \sum_{n=1}^{\infty} T_n R_n$$

$$= K_0 \ln\left(\frac{r}{b}\right) + \sum_{n=1}^{\infty} \left(d_{1n} r^n + d_{2n} r^{-n}\right)\left(c_{1n}\cos(n\theta) + c_{2n}\sin(n\theta)\right)$$

$$= K_0 \ln\left(\frac{r}{b}\right) + \sum_{n=1}^{\infty} \left(r^n + \tilde{d}_{2n} r^{-n}\right)\left(K_{1n}\cos(n\theta) + K_{2n}\sin(n\theta)\right)$$

Step 5: Apply remaining BC

$$u(a,\theta) = f(\theta) \Leftrightarrow K_0 \ln\left(\frac{a}{b}\right) + \sum_{n=1}^{\infty} \left(a^n + \tilde{d}_{2n} a^{-n}\right)\left(K_{1n}\cos(n\theta) + K_{2n}\sin(n\theta)\right) = f(\theta)$$

$$\frac{A_0}{2}$$

Euler formulas:

$$\begin{cases} A_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\, d\theta \\ A_n = K_{1n}\left(a^n + \tilde{d}_{2n} a^{-n}\right) = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta \\ B_n = K_{2n}\left(a^n + \tilde{d}_{2n} a^{-n}\right) = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta \end{cases}$$

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The Wave Equation: (Section 10.6)

* If $u(x,t)$ represents the displacement (deflection) of the string and the ends of the string are held fixed, then the motion of the string is governed by the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad 0<x<L,\ t>0$$

$$u(0,t) = u(L,t) = 0, \qquad t>0,$$

initial displacement $\rightarrow$ $$u(x,0) = f(x), \qquad 0<x<L,$$

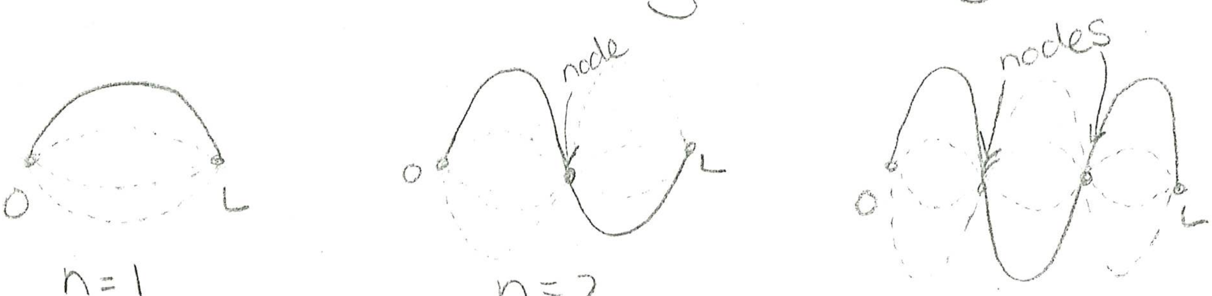
initial velocity $\rightarrow$ $$\frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0<x<L.$$

* Note α depends on the physical parameters of the string.

* Using the method of separation of variables, we found that

$$u_n(x,t) = \underbrace{\left[a_n \cos\left(\frac{n\pi\alpha t}{L}\right) + b_n \sin\left(\frac{n\pi\alpha t}{L}\right)\right]}_{\text{time-varying amplitude}} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{sinusoidal curve}}.$$

* For each value of n, $u_n(x,t)$ can be viewed as a <u>standing wave</u> (a wave that vibrates in place without lateral motion along the string).



$n=1$ $\qquad$ $n=2$ $\qquad$ $n=3$

* These are standing waves. The dashed curves show the time-varying amplitude

* For the n^{th} term, we have a sinusoid $\sin\left(\frac{n\pi x}{L}\right)$ with time-varying amplitude and $(n-1)$ nodes.
* The fundamental frequency is the frequency of the lowest mode ($u_n(x,t)$ for $n=1$), and integer multiples of the fundamental frequency are called harmonics.

* The eigenfrequencies of the vibrating string are the harmonics

$$\omega_n = \frac{n\pi\alpha}{L} \qquad \text{for } n=1,2,3,\ldots$$

* The general solution to the wave equation is expressed as a superposition of infinitely many standing waves.

Non-homogeneous Wave Equation Problem

Ex 1 The general case

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + h(x,t) \qquad 0<x<L,\ t>0$$

$$u(0,t) = u(L,t) = 0 \qquad t>0$$

$$u(x,0) = f(x) \qquad 0<x<L$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

Solution:

* Recall that for the homogeneous PDE

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

with conditions $u(0,t)=u(L,t)=0$, we expect a solution that consists of a superposition of standing waves

$$u_h(x,t)=\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi\alpha t}{L}\right)+b_n\sin\left(\frac{n\pi\alpha t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

* Motivated by this fact, let's try to find a solution to our non-homogeneous problem of the form

$$u(x,t)=\sum_{n=1}^{\infty}u_n(t)\sin\left(\frac{n\pi x}{L}\right), \qquad \cdots (1)$$

where the $u_n(t)$'s are functions of t to be determined.

* Note that if $h(x,t)$ is well-behaved, then for each fixed t, we can compute a Fourier Sine series for $h(x,t)$.

$$h(x,t)=\sum_{n=1}^{\infty}h_n(t)\sin\left(\frac{n\pi x}{L}\right). \qquad \cdots (2)$$

Since t is fixed, $h_n(t)=$ constant, so it is the coefficient in our Fourier Sine Series.

$$h_n(t)=\frac{2}{L}\int_0^L h(x,t)\sin\left(\frac{n\pi x}{L}\right)dx \qquad \text{for } n=1,2,\ldots$$

If all the above series converge, then substitution into the PDE yields

$$\frac{\partial^2 u}{\partial t^2}-\alpha^2\frac{\partial^2 u}{\partial x^2}=h(x,t)$$

$$\sum_{n=1}^{\infty}u_n''(t)\sin\left(\frac{n\pi x}{L}\right)-\alpha^2\sum_{n=1}^{\infty}u_n(t)\left(\frac{n\pi}{L}\right)^2\sin\left(\frac{n\pi x}{L}\right)=\sum_{n=1}^{\infty}h_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

$$\underbrace{\sum_{n=1}^{\infty}\left[u_n''(t)+\left(\frac{n\pi}{L}\alpha\right)^2u_n(t)\right]}\sin\left(\frac{n\pi x}{L}\right)=\underbrace{\sum_{n=1}^{\infty}h_n(t)}\sin\left(\frac{n\pi x}{L}\right)$$

$$= $$

Equating the coefficients in the series yields

$$u_n''(t) + \left(\frac{n\pi\alpha}{L}\right)^2 u_n(t) = h_n(t)$$

* This is a non-homogeneous, constant-coefficient equation that can be solved using the method of

Variation of Parameters

(This method is described in detail in Section 4.6)

The Method of Variation of Parameters:

Consider the non-homogeneous linear second-order equation

$$u_n''(t) + \left(\frac{n\pi\alpha}{L}\right)^2 u_n(t) = h_n(t) \quad \text{--- (1)}$$

and let $u_1(t)$ and $u_2(t)$ be two linearly independent solutions for the corresponding homogeneous equation

$$u_n''(t) + \left(\frac{n\pi\alpha}{L}\right)^2 u_n(t) = 0.$$

We know how to solve this homogeneous problem.

Let

$$u_n(t) = e^{rt}$$
$$u_n'(t) = re^{rt}$$
$$u_n''(t) = r^2e^{rt}$$

then

$$r^2e^{rt} + \left(\frac{n\pi\alpha}{L}\right)^2 e^{rt} = 0$$

$$\left(r^2 + \left(\frac{n\pi\alpha}{L}\right)^2\right) e^{rt} = 0$$

$$r^2 = -\left(\frac{n\pi\alpha}{L}\right)^2$$

$$r = \pm i\,\frac{n\pi\alpha}{L}$$ ← Imaginary values, so expect cosines & sines.

∴ The general solution to this homogeneous equation is

$$u_{n,h}(t) = a_n u_1(t) + b_n u_2(t)$$

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$$u(x,0) = \sum_{n=1}^{\infty} u_n(0) \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$0<x<L$$

$$u_t(x,0) = \sum_{n=1}^{\infty} u_n'(0) \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

Let's compute $u_n(0)$ & $u_n'(0)$.

$$\therefore\ u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \qquad 0<x<L$$

So

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx} \quad \text{----} (6)$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \underbrace{\left(\frac{n\pi\alpha}{L}\right)}_{\text{constant}} \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

So

$$b_n \frac{n\pi\alpha}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow \boxed{b_n = \frac{2}{n\pi\alpha}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx} \quad \text{---} (7)$$

Thus, the formal solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi\alpha}{L}t\right) + b_n\sin\left(\frac{n\pi\alpha}{L}t\right) + \frac{L}{n\pi\alpha}\int_0^t \sin\left(\frac{n\pi\alpha}{L}(t-s)\right)h_n(s)\,ds\right]\sin\left(\frac{n\pi x}{L}\right),$$

with the coefficients a_n & b_n as given by (6) & (7) respectively.