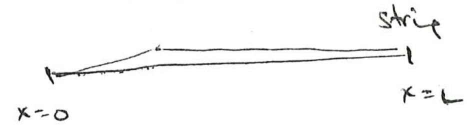
PDE - Another Example

Vibrating String

→ Simpler case



- stretch a string from $x=0$ to $x=L$
- pluck it → vibrates

Find a function that describes the vertical displacement of the string at any point x $\quad 0 \le x \le L$ and at any time $t>0$

$$y = u(x,t)$$

The motion of the string is described by the 1-dim^l wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

c - depends on the physical parameters of the string (mass, tension, etc...)

simplest solution $\quad u(x,t) = \sin\frac{\pi x}{L} \cos\frac{\pi c t}{L}$ $\quad$ "standing wave"

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verify $\quad u_t = \sin\frac{\pi x}{L} \cdot -\sin\frac{\pi c t}{L} \cdot \frac{\pi c}{L}$

$$u_{tt} = -\sin\frac{\pi x}{L}\left(\frac{\pi c}{L}\right)^2 \cos\frac{\pi c t}{L}$$

$$u_x = \cos\frac{\pi x}{L} \cdot \frac{\pi}{L} \cos\frac{\pi c t}{L}$$

$$u_{xx} = -\sin\frac{\pi x}{L}\left(\frac{\pi}{L}\right)^2 \cos\frac{\pi c t}{L}$$

$$u_{tt} = c^2 u_{xx} = -\frac{c^2\pi^2}{L^2} \sin\frac{\pi x}{L} \cos\frac{\pi c t}{L} \quad \checkmark$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ $\Rightarrow$ for each derivative, there is 1 degree of freedom.

Boundary conditions.

$\Rightarrow$

$u(0,t) = 0$
$u(L,t) = 0$ for all $t > 0$ physical characteristics

$u(x,0) = f(x)$ initial position

$\frac{\partial u}{\partial t}(x,0) = g(x)$ initial velocity

} for $0 < x < L$

Eg. $u(x,0) = \sin \frac{\pi x}{L}$ $\xrightarrow{\text{the arc of a sinusoidal curve}}$

$u_t(x,0) = 0$

$\Rightarrow$ pluck & drop
no initial velocity
or start from rest

~~Note~~ this corresponds to $\boxed{u(x,t) = \sin \frac{\pi x}{L} \cos \frac{\pi c t}{L}}$

check: at $t=0$ $u(x,0) = \sin \frac{\pi x}{L}$

$$\frac{\partial u}{\partial t} = -\frac{\pi c}{L} \sin \frac{\pi x}{L} \sin \frac{\pi c t}{L}$$ at $t=0$ $\Rightarrow \frac{\partial u}{\partial t} = 0$.

there are many solution to the general problem.

For any value of n

$$\sin \frac{n\pi x}{L} \cos \frac{n\pi c t}{L} \qquad n = 1, 2, 3, \ldots$$

or

$$\sin \frac{n\pi x}{L} \sin \frac{n\pi c t}{L}$$

go * first

We use the superposition principle to get a general solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \sin \frac{n\pi c t}{L} + b_n \cos \frac{n\pi c t}{L} \right) \sin \frac{n\pi x}{L}$$

* Consider $u_n(x,t) = \underbrace{\left(a_n \sin\frac{n\pi c t}{L} + b_n \cos\frac{n\pi c t}{L}\right)}_{\text{time varying amplitude.}} \sin\frac{n\pi x}{L}$

$\Downarrow$ sinusoidal curve. in x

The combination $u(x,t) = \sum_n$

allows us to describe more complicated string motion
string motion

note that given the boundary condition

$u(x,0) = f(x)$

$$u(x,0) = \boxed{\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = f(x)}$$ * function describing initial displacement

$\Downarrow$

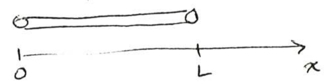
Fourier series representation for $F(x)$ → later!

~~Chpt. 2. Method of Separation of Variables~~

Chpt 10.2 Methods of Separation of Variables.

write equation first
Then give a possible physical description

Ex1 Consider



Heat flow model

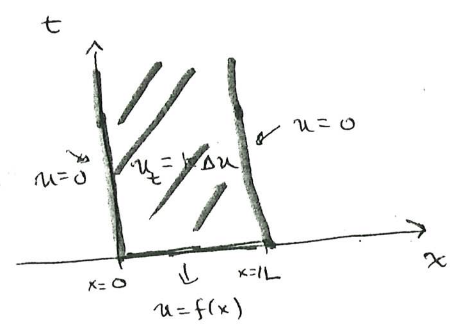
for heat through a thin insulated wire whose ends are kept at a constant temperature of 0°C with a specified initial temperature. ~~Assume the thermal coefficients are constant and there is no source of thermal energy~~.

"Initial Value Problem" (Boundary)

PDE: $\dfrac{\partial u}{\partial t} = K \dfrac{\partial^2 u}{\partial x^2}$ $\quad 0 < x < L$, $t > 0$

IC: $u(x,0) = f(x)$ $\quad 0 < x < L$

BC: $u(0,t) = 0$, $u(L,t) = 0$ $\quad \} \; t > 0$



"Method of Separation of Variables"

Assume the solution is of the form $\quad u(x,t) = X(x)\,T(t)$ $\quad$ i.e. assume the variables can be separated

∴ then $\dfrac{\partial u}{\partial t} = X(x)\dfrac{dT}{dt}$

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{d^2 X}{dx^2}\,T(t)$

Substitute in the PDE → ODE

$$X(x)\frac{dT}{dt} = K\frac{d^2X}{dx^2}T(t)$$

separate the variables

$$\boxed{\frac{1}{T}\frac{dT}{dt} = K\frac{1}{X}\frac{d^2X}{dx^2}}$$

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For convenience rewrite as

1st order, 2nd order

$$\boxed{\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}}$$

only depends on t

only depends on x

* If we vary x, then the left side does not vary. So the right side cannot vary either.
Same is true if you vary t, since the right side doesn't depend on t, there is no change so the expression on both sides are constant.

Both sides are constant

$$\boxed{\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda} \quad *$$

where λ is a real arbitrary constant known as the "separation constant"

$\Downarrow$

we choose to have a negative sign because ...

2 Equations

(1) $\frac{d^2X}{dx^2} = -\lambda$. 2nd Order ODE

(2) $\frac{dT}{dt} = -\lambda k$. 1st-order ODE

* Note that λ is the same constant in both (1) & (2) follows from *

What about the boundary conditions??

recall $u(x,t) = X(x)\,T(t)$

but. ① $u(0,t) = u(L,t) = 0$ for $t>0$ $\Rightarrow$ $X(0)T(t) = 0$, $X(L)\,T(t) = 0$

either $T(t) = 0 \quad \forall t>0$

$\Rightarrow u(x,t) = 0 \quad \forall t>0$

this is the trivial solution

the non-trivial solution is obtained from

or $X(0) = X(L) = 0 \quad \forall t$

Boundary-value Problem

$$\boxed{\begin{aligned} &\text{Ⓐ} \quad \frac{d^2X}{dx^2} = -\lambda X \\ &X(0) = 0 \\ &X(L) = 0 \end{aligned}}$$

time-dependent Equation

$$\text{Ⓑ} \quad \boxed{\frac{dT}{dt} = -\lambda k\, T}$$

* at $t=0$, $u(x,0) = X(x)T(0) = f(x)$

not a condition on $T(t)$

Ⓐ. $X'' + \lambda X = 0$

$X(0) = X(L) = 0$

2nd order ODE with constant coefficients

set $X(x) = e^{rx}$

$X'(x) = re^{rx}$

$X''(x) = r^2 e^{rx}$

$\Leftarrow$

$r^2 e^{rx} + \lambda e^{rx} = 0$

$(r^2+\lambda)e^{rx} = 0$

$r^2 + \lambda = 0$

$r^2 = -\lambda$

cases - $\lambda > 0$ then $r = \pm i\sqrt{\lambda}$
Two roots are pure imaginary

$\lambda = 0$ $r = 0$ is a double root

$\lambda < 0$ $r = \pm\sqrt{-\lambda}$ Two roots one is + the other is −

λ itself is complex

we always ignore this one

in fact later on we will be able to prove

that λ must be real if there is a

[illegible]

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We look at each case separately.

The solution oscillate

these are more convenient

then $e^{i\sqrt{\lambda}x}$, $e^{-i\sqrt{\lambda}x}$

Case 1

Say $\lambda > 0$ then $r = \pm i\sqrt{\lambda}$

so $\phi(x) = e^{\pm i\sqrt{\lambda} x}$

the general solution is $X(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

i.e a linear combination of two independent solution $\cos\sqrt{\lambda}\, x$ & $\sin\sqrt{\lambda}\, x$

Apply the Boundary Condition

$X(0) = 0$
$X(L) = 0$

① $X(0) = 0$

$0 = c_1$. so. $X(x) = c_2 \sin\sqrt{\lambda}\, x$

② $X(L) = 0$

$c_2 \sin\sqrt{\lambda}\, L = 0$

either $c_2 = 0$ $\Rightarrow$ trivial solution

or $\sin\sqrt{\lambda}\, L = 0$.

$\sqrt{\lambda}\, L = n\pi$ $\quad n = 1, 2, 3, \ldots$

$\sqrt{\lambda} = \frac{n\pi}{L}$

or $\lambda = \left(\frac{n\pi}{L}\right)^2$

∴ For each value of n,

we have $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ ← eigenvalues

a solution

$X_n(x) = C_n \sin\frac{n\pi}{L} x$

↖ eigenfunctions