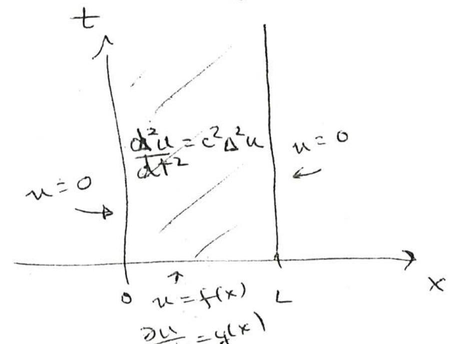
§ 10.2 -

Ex. 2. Solution for the 1-dim^l Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L, \quad t > 0$$



B.C. $u(0,t) = 0$ $\quad \forall\, t > 0$
$u(L,t) = 0$

I.C. $\boxed{u(x,0) = f(x)}$
$\frac{\partial u}{\partial t}(x,0) = g(x)$ $\quad 0 < x < L$ $\quad$ initial position & velocity

Step 1 done in PLH3

Step 1 → Separation of Variables

Assume $u = u(x,t) = X(x)T(t)$

Subs. in P.D.E.

STEP 1

$$\frac{X T''}{c^2 XT} = \frac{c^2 X'' T}{c^2 XT}$$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -\lambda$$

function of t — function of x

?? Note that the B.C. in x are homogeneous
* → This means this is where we will find our eigenvalues.

Step 2

$$\frac{G''}{c^2 G} = -\lambda \qquad \frac{F''}{F} = -\lambda$$

(1) $X'' + \lambda X = 0$

(2) $T'' + \lambda c^2 T = 0$
$u(x,0) = F(x)G(0) = 0$
$\therefore G(0) = 0$

B.C. $u(0,t) = X(0)T(t) = 0 \quad \forall t \Rightarrow X(0) = 0$
$u(L,t) = X(L)T(t) = 0 \quad \forall t \Rightarrow X(L) = 0$

* The condition in t are not homog. eqs
$= 0$ $X(x)T(0) = 0$
$u(x,0) = X(x)T(0) = f(x)$
these cannot be separated.

* Only homogeneous B.C can be separated *

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(1) $X(x)$; $X'' + \lambda X = 0$

$X(0) = X'(L) = 0$

(B)(2) $T(t)$; $T'' + \lambda c^2 T = 0$

$T(0) = 0$

Step 3 ~~[illegible]~~ Solve the separated equation

(4) 2nd-order ODE with constant coefficients

(B)

set $X(x) = e^{rx}$,

$X' = re^{rx}$

$X'' = r^2 e^{rx}$

$\Rightarrow r^2 e^{rx} + \lambda e^{rx} = 0$

$e^{rx}(r^2 + \lambda) = 0$

$r^2 = -\lambda$

$r = \pm\sqrt{-\lambda}$

3 cases.

(a) $\lambda = 0$

(b) $\lambda < 0$ $\quad \lambda = -\rho^2$, $r = \pm\rho$ ~~$\Rightarrow -\lambda = s > 0 \Rightarrow -\lambda = s^2 > 0$~~

(c) $\lambda > 0$ $\quad \lambda = \rho^2$, $r = \pm i\rho$

(a) $\lambda = 0 \Rightarrow X'' = 0$

$X' = c_1$

$X = c_1 x + c_2$

$X(0) = 0 \Rightarrow c_2 = 0$

$X(L) = 0 \Rightarrow c_1 L = 0$

only get the trivial solution

(b) $\lambda < 0 \Rightarrow$ $r = \pm s$ ~~$r = \pm\sqrt{s}$~~ real roots

$\Rightarrow X(x) \to e^{\sqrt{s}x}, e^{-\sqrt{s}x}$

$\to e^{sx}, e^{-sx}$

$\Rightarrow$ ~~$\cosh\sqrt{s}x, \sinh\sqrt{s}x$~~

$\to \cosh sx, \sinh sx$

$X(x) = c_1 \cosh sx + c_2 \sinh sx$

$X(0) = 0 = c_1$

$X(L) = 0 = c_2 \sinh sL = 0$

but $\sinh x = 0$ only at $x = 0$

$sL = 0 \Rightarrow s = 0$

only get the trivial solution

(c) $\lambda > 0 \Rightarrow$ say $\lambda = \mu^2 > 0$

then

$$r = \pm\sqrt{-\mu^2} = \pm i\mu .$$ complex roots

the general solution is given by

$$X(x) = c_1 \cancel{\cos \mu x} + c_2 \sin \mu x$$

$$X(0) = 0 = c_1$$

$$X(L) = 0 = c_2 \sin \mu L = 0$$

$$\mu_n L = n\pi \qquad n = 1, 2, \dots$$

$$\mu_n = \frac{n\pi}{L}$$

Solution To (A) (A) $\boxed{X_n(x) = c_n \sin \frac{n\pi}{L} x}$

$$\boxed{\lambda_n = \mu_n^2 = \frac{n^2\pi^2}{L^2} \qquad n = 1, 2, \dots}$$

(b) (B) $T'' + \lambda_n c^2 T = 0 \Rightarrow \lambda_n$ is determined.

$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0 \Rightarrow$ look for characteristic roots.

$r^2 = -\left(\frac{n\pi c}{L}\right)^2$ general solution

$$\longrightarrow T_n(t) = a_n \sin\left(\frac{n\pi c}{L}\right)t + \cancel{b_n \cos\left(\frac{n\pi c}{L}\right)t}$$

Last B.C. $T(0) = 0 \Rightarrow b_n = 0$.

Step 4 Superposition Principle

$$u_n(x,t) = X_n(x) T_n(t) = \sin \frac{n\pi}{L} x \left(a_n \sin \frac{n\pi c}{L} t + b_n \cos \left(\frac{n\pi c}{L}\right) t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \underbrace{\left(a_n \sin \frac{n\pi c}{L} t + \cancel{b_n \cos \frac{n\pi c}{L} t}\right)}_{\text{Time varying amplitude}} \sin \frac{n\pi}{L} x$$

sine wave.

$\boxed{\text{STEP 5}}$: Apply Remaining conditions

$\cancel{u(x,0) = f(x)}$ $\qquad 0 < x < L$

$u_t(x,0) = g(x)$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

this ~~will~~ condition determine the b_n's

next. $$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \sin \frac{n\pi c t}{L} + \cancel{b_n \cos \frac{n\pi c}{L} t} \right) \sin \frac{n\pi x}{L}$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(a_n \frac{n\pi c}{L} \cos \frac{n\pi c t}{L} + \cancel{b_n \frac{n\pi c}{L} \sin \frac{n\pi c}{L} t} \right) \sin \frac{n\pi x}{L}$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} a_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L}$$

$$g(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

A Fourier series representation for $g(x)$

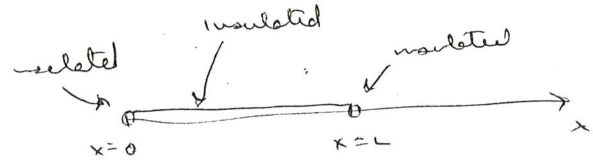
to here on Sep 11 (Lect #3)

Separation of Variables - Solution Steps

	Ex 1	Ex 2
PDE	$u_t = k u_{xx}$ Dir. BCs: $u(0,t) = u(L,t) = 0$ IC: $u(x,0) = f(x)$	$u_{tt} = c^2 u_{xx}$ Dir. BCs: $u(0,t) = u(L,t) = 0$ ICs: $u(x,0) = 0$ $u_t(x,0) = g(x)$
Step 1 Separate $u = F(x)G(t)$	$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$	$\frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda$
Step 2 Set up ODE problems	$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$ (A) $T' + \lambda k T = 0$ (B)	$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$ (A) $\begin{cases} T'' + c^2 \lambda T = 0 \\ T(0) = 0 \end{cases}$ (B)
Step 3 Solve ODE problems	(A) $\lambda = 0 \rightarrow$ trivial sol'ns $\lambda < 0 \rightarrow$ trivial sol'ns $\lambda > 0$, write $\lambda = \omega^2$ $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ eigenvalues $X_n(x) = c_n \sin\left(\frac{n\pi x}{L}\right)$ e-fu (B) $T_n(t) = a_n \exp\left(-\left(\frac{n\pi}{L}\right)^2 kt\right)$	(A) (same as in Ex 1) (B) $T_n(t) = a_n \sin\left(\frac{cn\pi t}{L}\right)$
Step 4 Form $u(x,t)$ (superposition principle)	$u_n(x,t) = X_n(x) T_n(t)$ $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi x}{L}\right)$	$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{cn\pi t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$
Step 5 Apply the remaining IC	$u(x,0) = f(x)$ $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$	$u_t(x,0) = g(x)$ $g(x) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} b_n \sin\left(\frac{n\pi x}{L}\right)$

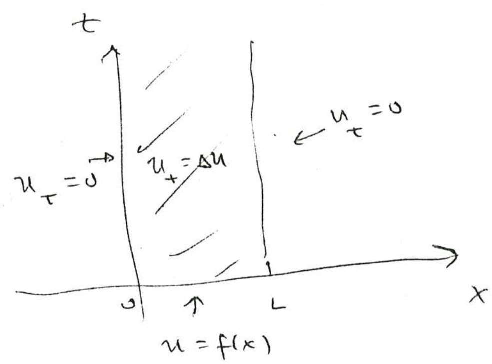
Bars with insulated ends

Ex 3 $\qquad \dfrac{\partial u}{\partial t} = \dfrac{c^2 \partial^2 u}{\partial x^2} \qquad 0 < x < L, \quad t > 0$



$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \qquad \text{Neumann type B.C.}$$

$$u(x,0) = f(x) \qquad \text{initial temperature}$$



Since the B.C. are homogeneous in x, we use the method of separation of variables

$\boxed{\text{Step 1:}}$ $\qquad u(x,t) = X(x)\,T(t)$

$$XT' = c^2 X'' T$$

$$\frac{1}{c^2}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

$\boxed{\text{Step 2:}}$ Get ODE's for X & T

① $X'' + \lambda X = 0$

$X'(0) = X'(L) = 0$

↓

must look at $\lambda = 0$, $\lambda < 0$, $\lambda > 0$

② $\dfrac{T'}{T} = -c^2\lambda \quad \to \quad T(t) = Ae^{-\lambda c^2 t}$

$\boxed{\text{Step 3}}$

Case 1) Ⓐ $\lambda = 0 \Rightarrow X'' = 0$

$X' = A$

$X = Ax + B$

$X' = A \to X'(0) = A = 0$

$X'(L) = 0$

so $X(x) = B$ is a solution

$$\boxed{\begin{array}{l}\lambda = 0 \text{ is an eigenvalue} \\ X_0(x) = B\end{array}}$$

→ when $\lambda = 0 \qquad T_0(t) = A$

so this is a solution $\boxed{\begin{array}{l} u = XT = A_0 \\ \lambda = 0\end{array}}$

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B case 2 $\lambda < 0$ say $\lambda = -\mu^2$ with $\mu > 0$

$$X'' - \mu^2 X = 0 \quad \Rightarrow \text{ set } X = e^{rx}$$

$$(r^2 - \mu^2) e^{rx} = 0 \quad \Rightarrow \quad r^2 = \mu^2$$

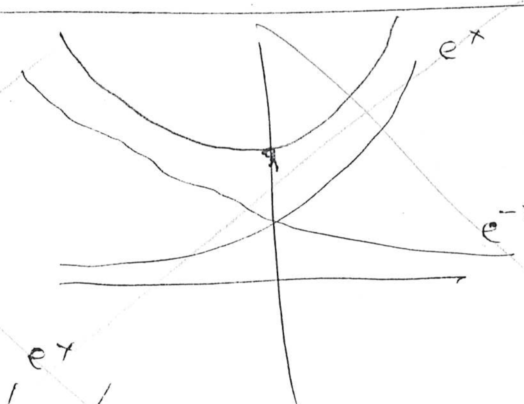
$$r = \pm \mu \quad \text{real roots}$$

either $X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$

or we use $\Rightarrow$ hyperbolic Functions.

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

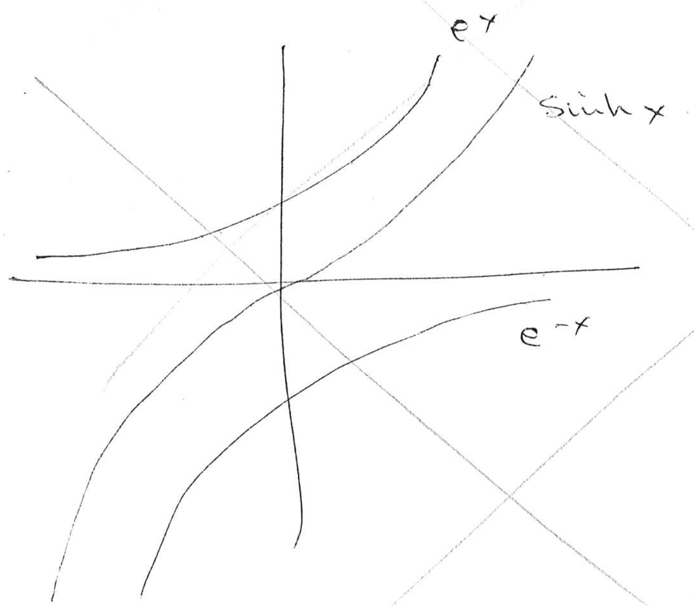
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



$\cosh x > 0 \quad \forall x$

$\cosh 0 = 1$

$* \quad \frac{d}{dx} \cosh x = \cosh x$



$\Rightarrow \sinh 0 = 0$

$\frac{d}{dx} \sinh x = \sinh x$

the general solution is

$$X(x) = c_1 \cosh \mu x + c_2 \sinh \mu x$$

$$X'(x) = c_1 \mu \sinh \mu x + c_2 \mu \cosh \mu x$$

$$X'(0) = c_2 \mu \quad \Rightarrow \mu \neq 0 \quad \Rightarrow c_2 = 0$$

$$X'(L) = \underbrace{c_1 \mu}_{\neq 0} \sinh \underbrace{\mu L}_{\neq 0} = 0 \quad \Rightarrow c_1 = 0$$

only get the trivial solution

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case 3) $\lambda > 0$ say $\lambda = \mu^2$ $\mu > 0$.

(A) $X'' + \mu^2 X = 0 \Rightarrow$ characteristic equation $r^2 = -\mu^2$

$r = \pm i\mu$.

complex roots

general solution is

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x$$

$$X'(x) = -\mu c_1 \sin \mu x + \mu c_2 \cos \mu x$$

$$X'(0) = 0 = \mu c_2 \quad \Rightarrow c_2 = 0$$

$$X'(L) = 0 = -\mu c_1 \sin \mu L = 0$$

$n = 0, 1, 2 \ldots$

$$\sin \mu L = 0 \iff \mu L = n\pi$$

$$\mu_n = \frac{n\pi}{L} \quad n = 1, 2, 3.$$

so $\lambda_n = \mu_n^2 = \frac{n^2\pi^2}{L^2}$ $n = 1, 2, 3$ are eigenvalues.

solution are

$$X_n(x) = C_n \cos \frac{n\pi}{L} x$$

* (B) (2) $\Rightarrow$ $T_n(t) = A_n e^{-\frac{n^2\pi^2 c^2}{L^2} t}$

$$\boxed{u_n(x,t) = B_n \cos \frac{n\pi}{L} x \, e^{-\frac{n^2\pi^2 c^2}{L^2} t} \quad n = 1, 2, \ldots}$$

STEP ~~3~~ 4.

Apply Principle of Superposition

$$u(x,t) = B_0 + \sum_{n=1}^{\infty} B_n \cos\frac{n\pi x}{L}\, e^{-\frac{n^2\pi^2 c^2 t}{L^2}}$$

note as $t \to \infty$

$u(x,t) \to B_0$.

STEP ~~4~~ 5.

At $t=0$

$$u(x,0) = f(x) = \underbrace{B_0 + \sum_{n=1}^{\infty} B_n \cos\frac{n\pi x}{L}}$$

* called a ~~half-range~~ cosine Fourier series for $f(x)$

Another Sep'n of Vars

§ 10.2 Example (if time - finish as much as possible)

Find the solution to the heat flow problem

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} \quad , 0 < x < \pi , \quad t > 0$$

$$u(0,t) = 0 , \quad \frac{\partial u}{\partial x}(\pi, t) = 0 \quad , t > 0$$

insulated end

$$u(x,0) = \sin(3x) + 5\sin(9x) - 2\sin(13x)$$

Sol'n:

Step 1: Separate

$$X(x)T'(t) = \gamma X''(x)T(t) \iff (1)$$

$$(1) \iff \frac{T'}{T} = \gamma \frac{X''}{X} \iff \frac{T'}{\gamma T} = \frac{X''}{X} = -\lambda$$

Step 2: ODEs

Then

$$\begin{cases} X'' + \lambda X = 0, & X(0) = 0, \ X'(\pi) = 0 \quad (BVP) \\ T' + \lambda \gamma T = 0 \end{cases}$$

Step 3: Solve the ODEs

① BVP - 1st ODE

Case 1: $\lambda < 0 = -\omega^2$

$$X'' + \lambda X = 0 \iff X'' - \omega^2 X = 0$$

char eqn: $r^2 - \omega^2 = 0 \iff r = \pm \omega$

$$\therefore X(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

Apply the BVs:

$$\begin{cases} X(0)=0 \\ X'(\pi)=0 \end{cases} \Leftrightarrow \begin{cases} c_1+c_2=0 \\ \omega c_1 e^{\omega\pi} + \omega c_2 e^{-\omega\pi} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = -c_2 \\ \omega c_2 \left(e^{-\omega\pi} - e^{\omega\pi}\right) = 0 \end{cases}$$

$e^{-\omega\pi} = e^{\omega\pi}$

$1 = e^{2\omega\pi} \Leftrightarrow \omega = 0$

$$\Leftrightarrow \begin{cases} c_1 = -c_2 \\ \omega = 0 \text{ or } c_2 = 0 \end{cases}$$

In either case, we arrive at the trivial solution $X(x)=0$.

Case 2: $\lambda = 0$

$X'' + \lambda X = 0 \Leftrightarrow X'' = 0 \Leftrightarrow X(x) = c_1 + c_2 x$

Apply the BVs:

$$\begin{cases} X(0)=0 \Leftrightarrow c_1 = 0 \\ X'(\pi)=0 \Leftrightarrow c_2 = 0 \end{cases}$$

$\therefore$ we arrive at the trivial solution, $X(x)=0$.

Case 3: $\lambda > 0 = \omega^2$

$$X'' + \lambda X = 0 \Leftrightarrow X'' + \omega^2 X = 0$$

char eqn: $r^2 + \omega^2 = 0 \Leftrightarrow r = \pm i\omega$

$$\therefore X(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$$

Apply BVs:

$$\begin{cases} X(0) = 0 \\ X'(\pi) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ \omega c_2 \cos(\omega \pi) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \text{ or } \omega = 0 \\ \text{or } \omega \pi = (2n+1)\frac{\pi}{2}, \; n \in \mathbb{Z} \end{cases}$$

eigenvalue

If $\boxed{\omega = \frac{2n+1}{2},}$ $n \in \mathbb{Z}$, then the BVP yields the nontrivial solution

$$\boxed{X_n(x) = c_2 \sin\left(\left(\frac{2n+1}{2}\right) x\right).}$$

eigenfunction (w/o the c_2)

② 2^{nd} ODE

For each value of ω, we have

$$T' + \omega^2 \gamma T = 0 \iff \boxed{T(t) = c_3 e^{-\omega^2 \gamma t}}$$

Step 4: Superposition

$$u(x,t) = \sum_n X_n(x) T_n(t)$$

$$= \sum_n a_n \sin\left(\left(\frac{2n+1}{2}\right)x\right) e^{-\left(\frac{2n+1}{2}\right)^2 \gamma t}$$

Step 5: Apply IC

$$u(x,0) = \sin(3x) + 5\sin(9x) - 2\sin(13x)$$

$$= \sum_n a_n \sin\left(\frac{(2n+1)}{2} x\right)$$

$\therefore \; a_1 = 1, \; a_4 = 5, \; a_6 = -2, \; a_n = 0 \;\; \forall n \neq 1,4,6$

$$\therefore \boxed{u(x,t) = \sin(3x) e^{-9\gamma t} + 5\sin(9x) e^{-81\gamma t} - 2\sin(13x) e^{-169\gamma t}}$$