§10.3 Fourier Series

Recall, Day 2

vibrating string:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L, \quad t > 0 \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u(x,0) = f(x) & 0 < x < L \end{cases}$$

Sol'n:

$$u(x,t) = \sum_n \left(a_n \sin\left(\frac{n\pi c t}{L}\right) + b_n \cos\left(\frac{n\pi c t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

Apply IC:

$$u(x,0) = \sum_n b_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

How to find b_n?

Other Series: Taylor Series

$$f(x) = f(0) + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

where $a_n = \dfrac{f^{(n)}(0)}{n!}$.

Fourier Series Expansions

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)}$$

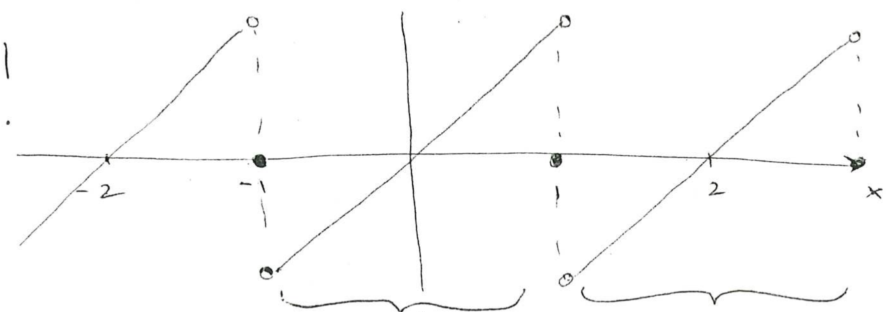
choice by our textbook
(makes formula for a_n easier to remember)

To find the formula for a_n + b_n, we need some theory... (this may take us two lectures ☺)

Defⁿ f is <u>periodic</u> of period $T > 0$ if $f(x) = f(x+T)$ for all x in the domain of f.

Ex. 1

Give formula.



period is 2

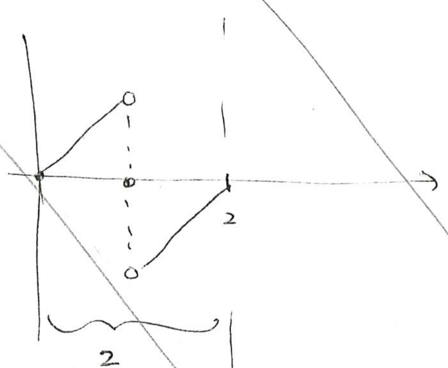
Describe the 2-periodic function f by considering its value on the interval $-1 < x \le 1$.

$$f(x) = \begin{cases} x & -1 < x < 1 \\ 0 & x = 1 \end{cases}$$

$f(x) = f(x+2)$ for all other values of x

periodic function
* need definition in $-T < 0 < T$
* extend using periodicity

~~Ex. 2 Consider f as above over the interval $0 \le x < 2$~~



period is 2

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ 0 & x = 1 \\ x-2 & 1 < x < 2 \end{cases}$$

$f(x) = f(x+2)$ otherwise

Note – $f(x)$ is not continuous at $x = \pm 1, \pm 2, \pm 3, \dots$

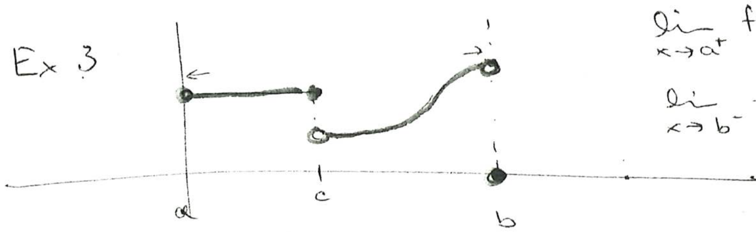
but it is relatively well behaved.

i.e. on the interval $0 \le x < 2$ there are only finitely many points of discontinuity.

also $\lim_{x \to 1^-} f(x) = 1 = f(1^-)$; $\lim_{x \to 1^+} f(x) = -1 = f(1^+)$ } left- & right-handed limit exists.

Def^n A function f is said to be <u>piecewise continuous</u> on $[a,b]$ if f is continuous at every point in $[a,b]$, except possibly for a finite number of points at which f has a jump discontinuity.

Ex 3



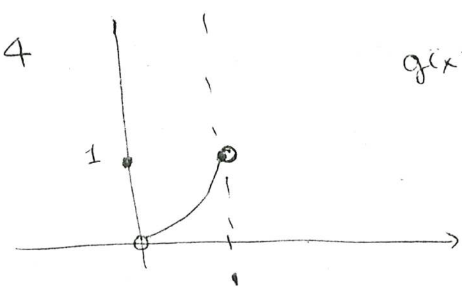
$\lim_{x\to a^+} f(x) = f(a^+)$

$\lim_{x\to b^-} f(x) = f(b^-)$ both exist

f is defined on $[a,b]$

2 jump discontinuities at c & b.

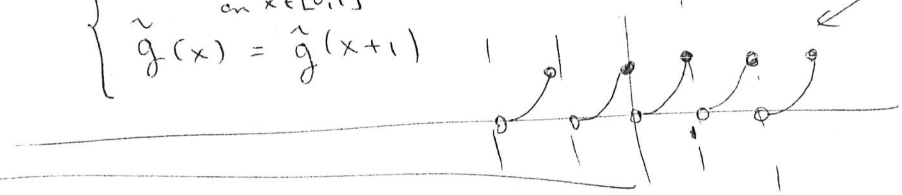
Ex. 4



$g(x)$ is piecewise continuous on $[0,1]$

note that we can extend $g(x)$ so that it is periodic with period $T=1$

$$\begin{cases} \tilde{g}(x) = g(x) & \text{on } x\in[0,1] \\ \tilde{g}(x) = \tilde{g}(x+1) \end{cases}$$



no longer symmetric about y axis

Fact the trigonometric functions $1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \ldots, \sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots$

denoted $\left\{1, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\right\}_{n=1}^{\infty}$

form an orthogonal set on $[-L, L]$

Def^n A set of functions $\{f_n(x)\}_{n=1}^{\infty}$ is an <u>orthogonal system</u> with respect to the weight function $w(x)$ on $[a,b]$ if

$$\underbrace{\int_a^b f_m(x) f_n(x) w(x)\,dx}_{} = 0 \quad \text{whenever } m \neq n,$$

↳ the integral corresponds to an inner product on the space of piecewise continuous functions on $[a,b]$.

*Note: like vectors & inner product.
$\vec{v}$ & $\vec{w}$ are orthogonal if $\vec{v}\cdot\vec{w} = 0$

Need to show that $\{1, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\}_{n=1}^{\infty}$ is orthogonal on $[-L,L]$.

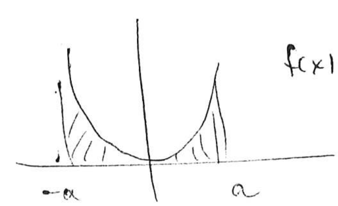
Consider. (1) $\displaystyle\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$

(2) $\displaystyle\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx$

(3) $\displaystyle\int_{-L}^{L} \cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx$

Facts about even & odd function

(A) $f(x)$ is even $\Leftrightarrow$ $f(-x) = f(x)$



$f(x) = x^2$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$$= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$$

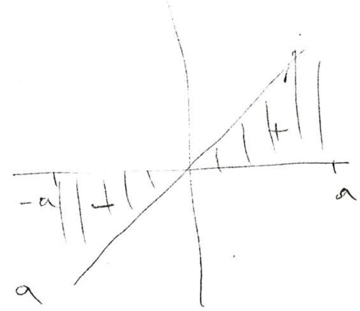
$-x = u$
$-dx = +du$
$x = 0 \Rightarrow u = 0$

$f(-u) = f(u)$

$$= \int_0^a f(-u)\,du + \int_0^a f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(u)\,du$$

$$= \int_0^a f(u)\,du$$

(B) f is odd $\iff$ $f(-x) = -f(x)$



$$\int_{-a}^{a} f(x)\,dx = 0$$

Similarly $\Rightarrow$ proof.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{a} f(x)\,dx \quad \ldots$$

$\cos\frac{n\pi x}{L}$ with $n=0$

1, $\cos\frac{n\pi x}{L}$ are even for $n = 1, \ldots$

$\sin\frac{n\pi x}{L}$ are odd for $n = 1, \ldots$

(1) $$\int_{-L}^{L} \underbrace{\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}}_{} dx = 2\int_{0}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$$

product of even funcs is even.

$f(-x)\,g(-x) = f(x)\,g(x)$

$$= \int_{0}^{L}\left[\cos(n-m)\frac{\pi x}{L} + \cos(n+m)\frac{\pi x}{L}\right]dx$$

if $n \neq m$ $\neq 0$

$$= \left[\frac{\sin(n-m)\frac{\pi x}{L}}{(n-m)\pi/L} + \frac{\sin(n+m)\frac{\pi x}{L}}{(n+m)\pi/L}\right]\Bigg|_{0}^{L}$$

Trig Identity

$2\cos\phi\cos\theta = \cos(\phi-\theta) + \cos(\phi+\theta)$

$\cos^2\phi = \frac{1+\cos 2\phi}{2}$

$$= \left(\frac{\overset{0}{\sin(n-m)\pi}}{(n-m)\pi/L} + \frac{\overset{0}{\sin(n+m)\pi}}{(n+m)\pi/L}\right) - (0+0)$$

$= 0$

if $n = m \neq 0$, $= 2\int_{0}^{L}\cos^2\frac{n\pi x}{L}\,dx = \int_{0}^{L}\left(1 + \cos\frac{2n\pi x}{L}\right)dx$

$$= \left(x + \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right)\Bigg|_{0}^{L} = L + \frac{L}{2n\pi}\left(\overset{0}{\sin 2n\pi} - 0\right)$$

-1

if $n = m = 0$ $\quad \int_{-L}^{L} 1\, dx = 2\int_{0}^{L} 1\, dx = 2x\Big|_0^L = 2L$

$$① \quad \int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{m\pi x}{L}\, dx = \begin{cases} 0, & m \neq n \\ L, & m = n \neq 0 \\ 2L, & m = n = 0 \end{cases}$$

$n = 0, 1, 2, \ldots$

$$② \quad \int_{-L}^{L} \underbrace{\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}}_{\text{product of odd functions}}\, dx = 2\int_{0}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\, dx$$

$$= \int_0^L \left[\cos(n-m)\frac{\pi x}{L} - \cos(n+m)\frac{\pi x}{L}\right] dx$$

$f(-x)g(-x) = -f(x)(-g(x)) = f(x)g(x)$

$\Rightarrow$ even

if $n \neq m$

$$= \left[\frac{\sin(n-m)\frac{\pi x}{L}}{(n-m)\pi/L} - \frac{\sin(n+m)\frac{\pi x}{L}}{(n+m)\pi/L}\right]\Bigg|_0^L$$

Trig Id.

$2\sin\phi\sin\theta = \cos(\phi-\theta) - \cos(\phi+\theta)$

$\sin^2\phi = \frac{1-\cos 2\phi}{2}$

$$= \left(\frac{\overset{0}{\sin(n-m)\pi}}{(n-m)\pi/L} - \frac{\overset{0}{\sin(n+m)\pi}}{(n+m)\pi/L}\right) - (0-0)$$

$$= 0$$

if $n = m \cancel{\neq 0}$

$$= 2\int_0^L \sin^2\frac{n\pi x}{L}\, dx$$

$$= \int_0^L \left(1-\cos\frac{2n\pi x}{L}\right) dx = \left(x - \frac{\sin\frac{2n\pi x}{L}}{\frac{2n\pi}{L}}\right)\Bigg|_0^L = L$$

if $n \cancel{= m} = 0$

$$② \quad \int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\, dx = \begin{cases} 0, & m \neq n, \\ L, & m = n. \end{cases}$$

$n = 1, 2, \ldots$

$$\boxed{③ \quad \int_{-L}^{L} \underbrace{\cos\frac{n\pi x}{L} \quad \sin\frac{m\pi x}{L}}_{\text{odd} \times \text{even}} \, dx = 0}$$

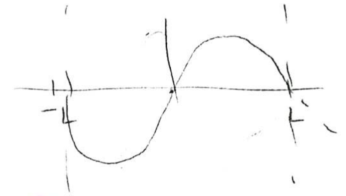
note
$$\int_{-L}^{L} 1 \cdot \cos\frac{n\pi x}{L}\, dx = 0$$
$$\int_{-L}^{L} 1 \cdot \sin\frac{n\pi x}{L}\, dx = 0$$

$f(-x)\,g(-x) = -f(x)\,g(x) \Rightarrow$ odd.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

* if 1, $\cos\frac{n\pi x}{L}$, $\sin\frac{n\pi x}{L}$ are periodic with period $2L$

($2L$ - periodic)



Then ~~f(x)~~ the sum must converge to a $2L$-periodic functi.

We use the fact that $\left\{1, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\right\}_{n=1}^{\infty}$ form an orthogonal set to solve for the coefficinets,

$$\int_{-L}^{L} f(x)\, dx = \int_{-L}^{L}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]\right) dx$$

$$2\int_{0}^{L} \cos\frac{n\pi x}{L}\, dx = \frac{2L}{n\pi} \sin\frac{n\pi x}{L}\Big|_0^L$$

$$= \int_{-L}^{L} \frac{a_0}{2}\, dx + \int_{-L}^{L} \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}\, dx + \int_{-L}^{L} \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{a_0}{2}\int_{-L}^{L} 1\, dx + \sum_{n=1}^{\infty} a_n \cancelto{0}{\int_{-L}^{L} 1 \cdot \cos\frac{n\pi x}{L}\, dx} + \sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{-L}^{L} 1 \cdot \sin\frac{n\pi x}{L}\, dx}$$

$$\int_{-L}^{L} f(x)\, dx = \frac{a_0}{2}(2L)$$

$$\Rightarrow \quad a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx$$

a_0 is the average of f on the interval $[-L, L]$.

(41)

next

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \int_{-L}^{L} a_0 \cos\frac{m\pi x}{L}\,dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$$

if $n \neq m$

$$= 0$$

---

if $n = m$

$$= a_m \int_{-L}^{L} \cos^2\frac{m\pi x}{L}\,dx$$

$$= a_m L$$

so
$$\boxed{a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx}$$

true for $n=0$

Likewise, we can show that

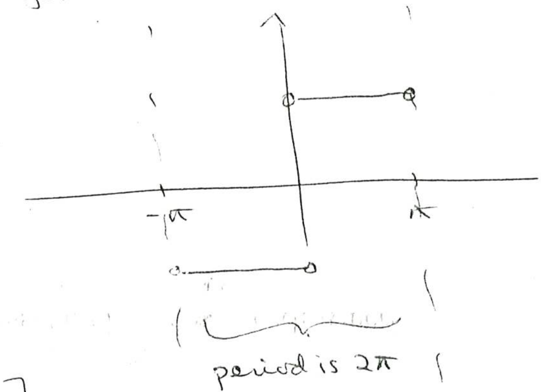
$$\boxed{b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx}$$

these are known as the Euler Formulas

+2

Ex. 1. Compute the Fourier Series for

$$f(x) = \begin{cases} -1 & -\pi < x < 0, \\ 1 & 0 < x < \pi. \end{cases}$$

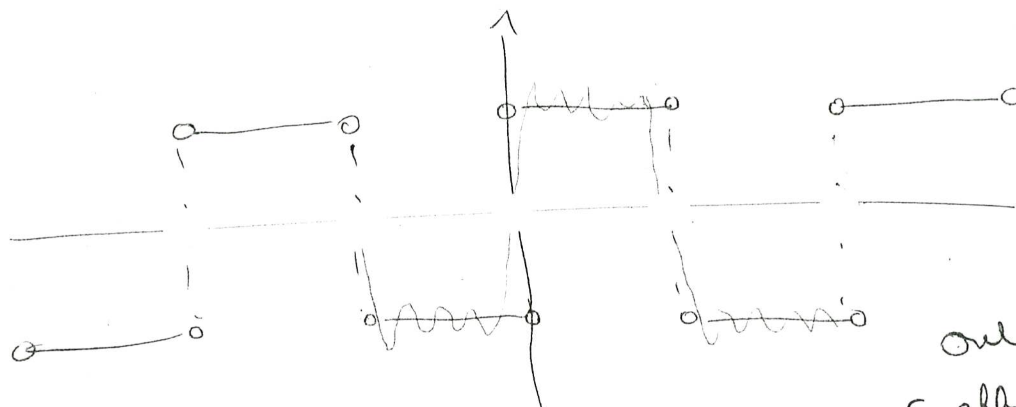


$L = \pi$

we write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]$$

the series (if it converges) converges to
a 2π periodic function $\hat{f}(x)$ *



Only do coefficient first

$$\tilde{f} = \begin{cases} -1 & -\pi < x < 0, \\ 1 & 0 < x < \pi \\ & , \ldots \end{cases}$$

$$\tilde{f}(x) = \tilde{f}(x + 2\pi)$$

* Add to first Assignment
use Maple to see how well the series converges

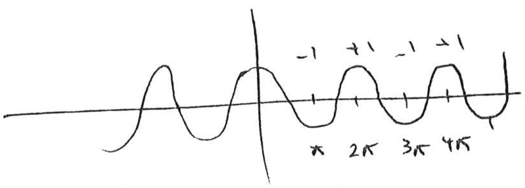
Compute the coefficient

for $n = 0, 1, 2, \ldots$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{odd}} \times \underbrace{\cos nx}_{\text{even}}}_{\text{odd on a symmetric interval}} \, dx = 0$$

(13)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin nx}_{\text{odd} \times \text{odd} = \text{even}}\,dx = \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx$$



$$= \frac{2}{\pi}\left[-\frac{\cos nx}{n} \Big|_0^{\pi} \right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n} \right] \qquad n = 1, 2, 3, \ldots$$

$\cos n\pi$

$$= \begin{cases} 0 & n \text{ is even} \\ \dfrac{4}{\pi n} & n \text{ is odd} \end{cases}$$

$$f(x) \sim \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin nx$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x \cdots \right.$$

* Fourier Series have a much wider range of application than solving P.D.E.

* Physical Geographer / Engineers

→ incoming signal ⇒ break down into its harmonic component ⇒ Fourier Representation

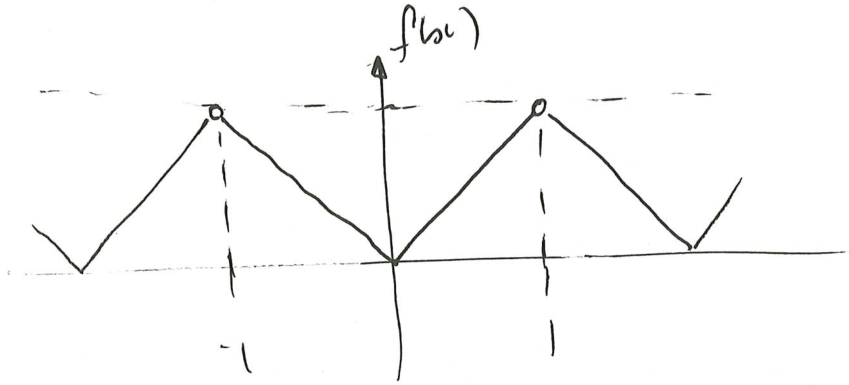
the frequency ↔ eigenvalues → audible tone or color spectrum

"Spectral Analysis"

Another example (ex 3 in text):

Compute the Fourier series for $f(x) = |x|$, $-1 \leq x \leq 1$

was $<$? why?



<u>Sol'n</u>

$f(x)$ is even, $\therefore$ $f(x)\sin(n\pi x)$ is odd, so

$$\int_{-1}^{1} f(x)\sin(n\pi x)\,dx = 0.$$

Similarly, $f(x)\cos(n\pi x)$ is even, so

$$a_0 \int_{-1}^{1} f(x)\,dx = 2\int_0^1 x\,dx = x^2\Big|_0^1 = 1$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = 2\int_0^1 x\cos(n\pi x)\,dx = \frac{2}{\pi^2 n^2}\left[(-1)^n - 1\right]$$

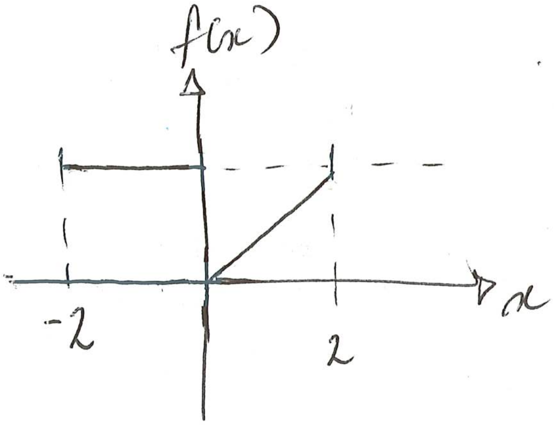
$n = 1, 2, 3, \ldots$

$$\therefore f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2}\left[(-1)^n - 1\right]\cos(n\pi x)$$

Ex 3

Compute the Fourier Series for

$$f(x) = \begin{cases} 1, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$$



Neither odd nor even.

$L = 2$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-2}^{0} 1\,dx + \frac{1}{2}\int_{0}^{2} x\,dx = \frac{1}{2}\left[2 + \frac{x^2}{2}\Big|_0^2\right] = \frac{2+2}{2} = 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-2}^{0}\cos\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Bigg|_{-2}^{0} + \frac{1}{2}\left[\frac{4}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= (0 - 0) + \frac{1}{2}\left[\frac{4}{(n\pi)^2}\left[(-1)^n - 1\right] + \frac{4}{n\pi}\cdot 0\right]$$

$$= \frac{2\left[(-1)^n - 1\right]}{(n\pi)^2}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-2}^{0}\sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{1}{2}\,\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_{-2}^{0} + \frac{1}{2}\left[\frac{-2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}x\right)\right]_{0}^{2}$$

$$= -\frac{1}{n\pi}\left[1-(-1)^n\right] + \left[-\frac{2}{n\pi}\left[(-1)^n-1\right] + \frac{2}{(n\pi)^2}\cdot 0\right]$$

$$= -\frac{1}{n\pi}\left[1-(-1)^n\right] + \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$= \frac{1}{n\pi}\left[1-(-1)^n\right]$$

$$\therefore f(x) = 1 + \sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\left[(-1)^n-1\right]\cos\left(\frac{n\pi x}{2}\right)$$

$$+ \sum_{n=1}^{\infty}\frac{1}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi x}{2}\right)$$