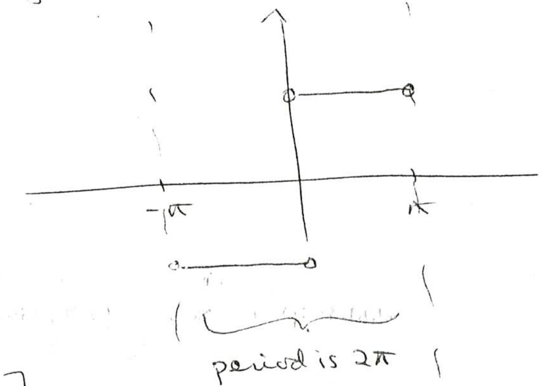
Ex. 1. Compute the Fourier Series for

$$f(x) = \begin{cases} -1 & -\pi < x < 0, \\ 1 & 0 < x < \pi. \end{cases}$$



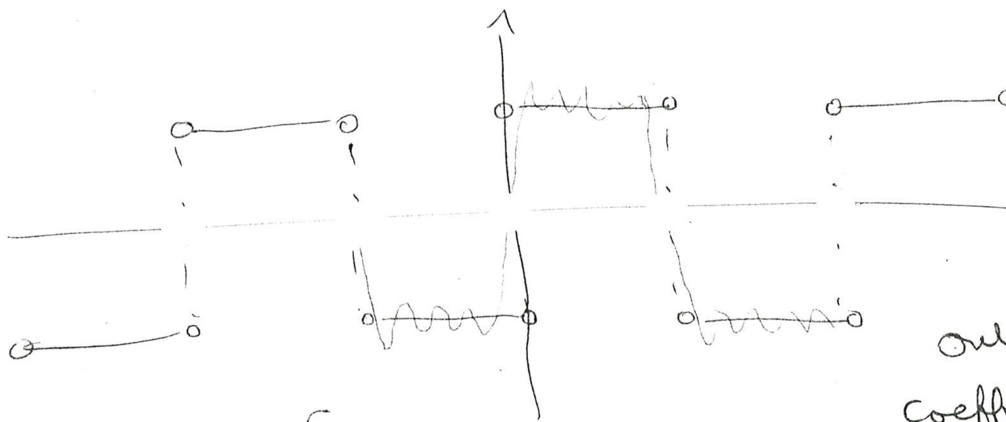
we write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]$$

$L = \pi$

the series (if it converges) converges to:

a 2π periodic funct. $\tilde{f}(x)$ *



Only do coefficient first

$$\tilde{f} = \begin{cases} -1 & -\pi < x < 0, \\ 1 & 0 < x < \pi \\ & , \ldots \end{cases}$$

$$\tilde{f}(x) = \tilde{f}(x + 2\pi)$$

Compute the coefficient

for $n = 0, 1, 2, \ldots$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{odd}} \underbrace{\cos nx}_{\text{even}}}_{\text{odd on a symmetric interval}} \, dx = 0$$

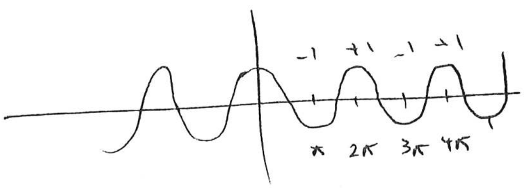
* Add to first Assignment

use Maple to see how well the series converges

(13)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx$$

$\underbrace{\text{odd} \times \text{odd}}_{\text{even}}$



$$= \frac{2}{\pi}\left[-\frac{\cos nx}{n}\Big|_0^{\pi} \right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n} \right] \qquad n = 1, 2, 3, \ldots$$

$\cos n\pi$

$$= \begin{cases} 0 & n \text{ is even} \\ \dfrac{4}{\pi n} & n \text{ is odd} \end{cases}$$

$$f(x) \sim \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n}\sin nx$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x \cdots \right.$$

* Fourier Series have a much wider range of application than solving P.D.E.

* Physical Geographer / Engineers

→ incoming signal ⇒ break down into its harmonic component ⇒ Fourier Representation

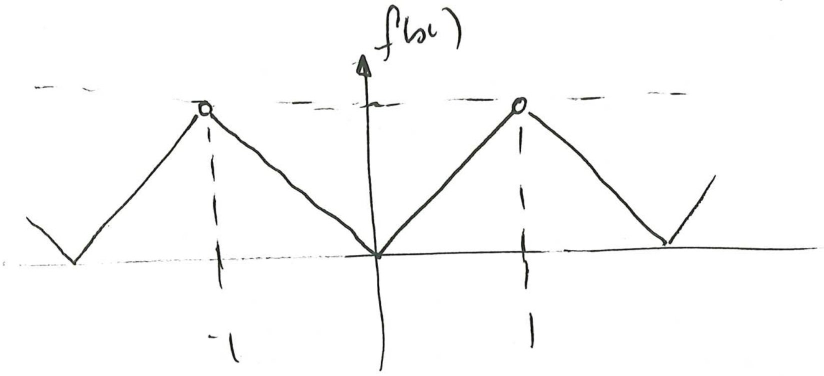
"Spectral Analysis"

the frequency ⟷ eigenvalues
↓
audible tone
or color spectrum.

Another example (ex 3 in text):

Compute the Fourier series for $f(x) = |x|$, $-1 \leq x \leq 1$

was $<$? why?



Sol'n

$f(x)$ is even, $\therefore$ $f(x)\sin(n\pi x)$ is odd, so

$$\int_{-1}^{1} f(x)\sin(n\pi x)\,dx = 0.$$

Similarly, $f(x)\cos(n\pi x)$ is even, so

$$a_0 \int_{-1}^{1} f(x)\,dx = 2\int_0^1 x\,dx = x^2\Big|_0^1 = 1$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = 2\int_0^1 x\cos(n\pi x)\,dx = \frac{2}{\pi^2 n^2}\left[(-1)^n - 1\right]$$

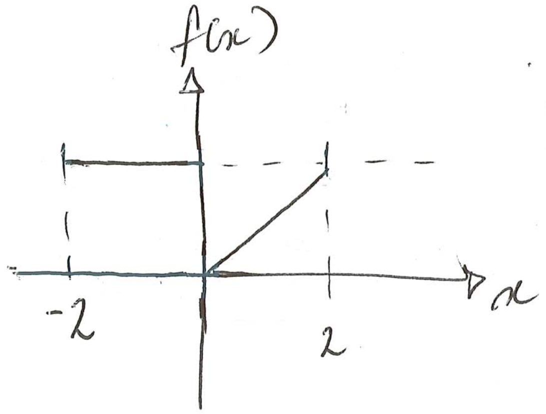
$n = 1, 2, 3, \ldots$

$$\therefore f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2}\left[(-1)^n - 1\right]\cos(n\pi x)$$

Ex 3

Compute the Fourier Series for

$$f(x) = \begin{cases} 1, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$$



Neither odd nor even.

$L = 2$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-2}^{0} 1\,dx + \frac{1}{2}\int_{0}^{2} x\,dx = \frac{1}{2}\left[2 + \frac{x^2}{2}\Big|_0^2\right] = \frac{2+2}{2} = 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-2}^{0}\cos\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{0} + \frac{1}{2}\left[\frac{4}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= (0-0) + \frac{1}{2}\left[\frac{4}{(n\pi)^2}\left[(-1)^n - 1\right] + \frac{4}{n\pi}\cdot 0\right]$$

$$= \frac{2\left[(-1)^n - 1\right]}{(n\pi)^2}$$

$b_n = \ldots$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-2}^{0}\sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{1}{2}\,\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{0} + \frac{1}{2}\left[\frac{-2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}x\right)\right]_{0}^{2}$$

$$= -\frac{1}{n\pi}\left[1-(-1)^n\right] + \left[-\frac{2}{n\pi}\left[(-1)^n-1\right] + \frac{2}{(n\pi)^2}\cdot 0\right]$$

$$= -\frac{1}{n\pi}\left[1-(-1)^n\right] + \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$= \frac{1}{n\pi}\left[1-(-1)^n\right]$$

$$\therefore f(x) = 1 + \sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\left[(-1)^n-1\right]\cos\left(\frac{n\pi x}{2}\right)$$

$$+ \sum_{n=1}^{\infty}\frac{1}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi x}{2}\right)$$

~~Check answer:~~

Convergence

Thm 2: Pointwise Cgce of Fourier Series

If f + f' are piecewise continuous on $[-L, L]$, then for any x in $(-L, L)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= \frac{1}{2}\left[f(x^+) + f(x^-)\right] \quad . \; . \; . \; . \; . \quad (6)$$

where the a_n + b_n values are given by the Euler Formulas (4) + (5). For $x = \pm L$, the series converges to

$$\frac{1}{2}\left[f(-L^+) + f(L^-)\right].$$

Ex To which fn does the Fourier series for

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

converge?