Convergence

Thm 2: Pointwise Cgce of Fourier Series

If f + f' are piecewise continuous on $[-L, L]$, then for any x in $(-L, L)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) = \frac{1}{2}\left[f(x^+) + f(x^-)\right] \quad \ldots\ldots \quad (6)$$

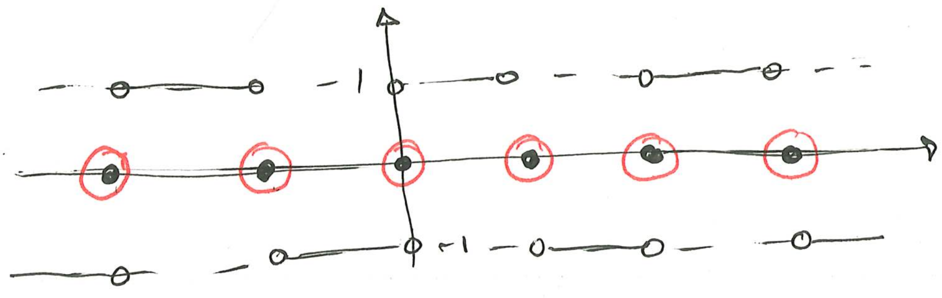
where the a_n + b_n values are given by the Euler Formulas (4) + (5). For $x = \pm L$, the series converges to

$$\frac{1}{2}\left[f(-L^+) + f(L^-)\right].$$

Ex To which fn does the Fourier Series for

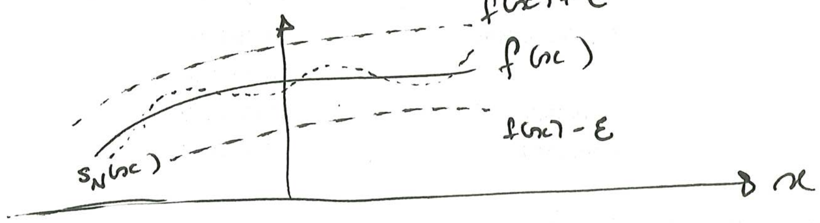
$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

converge?



If f is a $2L$-periodic fn that is continuous on $(-\infty, \infty)$ & has a piecewise continuous derivative, its Fourier series not only converges at each point, it converges uniformly on $(-\infty, \infty)$.

(illustrated by P2#5, prob 2)



Thm 3: Uniform Cgce of Fourier Series

Let f be a continuous function on $(-\infty, \infty)$ and periodic of period $2L$. If f' is piecewise continuous on $[-L, L]$, then the Fourier series for f converges uniformly to f on $[-L, L]$ and hence on any interval. That is, for each $\varepsilon > 0$, there exists an integer N_0 (that depends on ε) such that

$$\left| f(x) - \left(\frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \right) \right| < \varepsilon$$

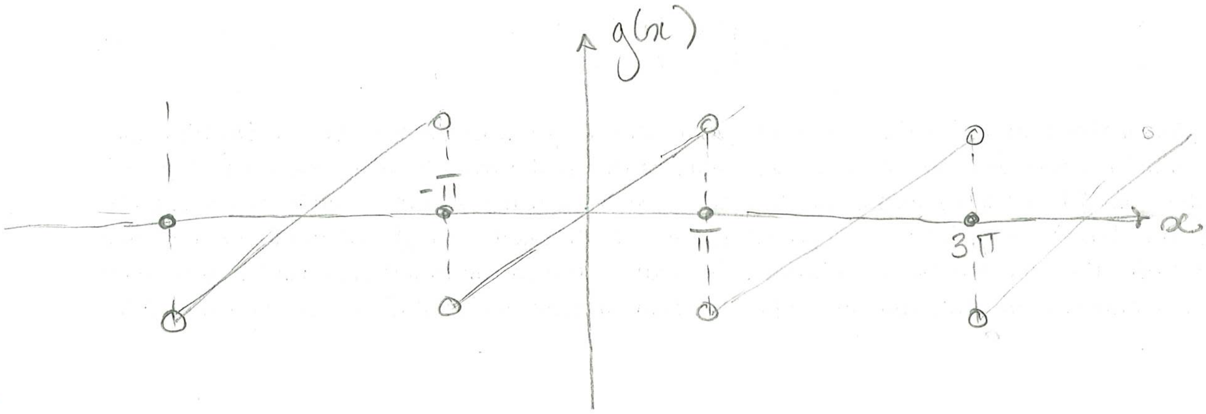
$\forall N \geq N_0$ & $\forall x \in (-\infty, \infty)$.

Note: can't always differentiate or integrate a Fourier series. See the text for details.

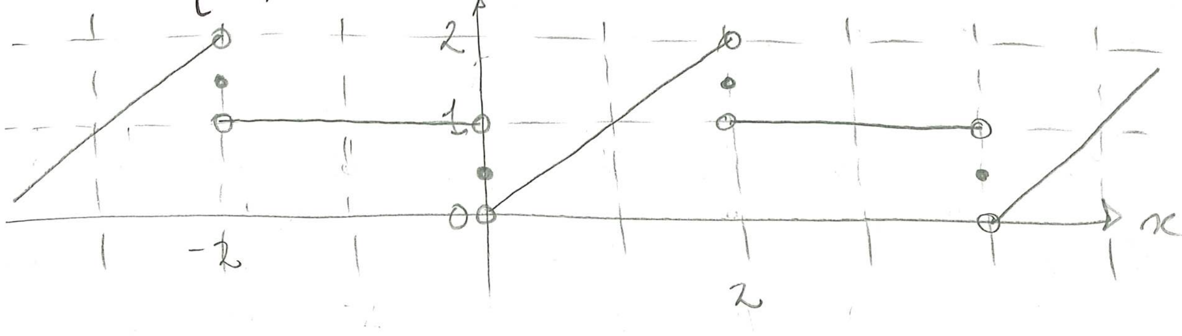
Examples

1) To what function $g(x)$ do the ~~following~~ Fourier Series for the following fns $f(x)$, converge?

a) $f(x) = x, \quad -\pi < x < \pi$



b) $f(x) = \begin{cases} 1, & -2 < x < 0, \\ x, & 0 < x < 2 \end{cases}$



So, the Fourier series we obtain ~~is~~ varies with the extension we choose. But, in every case, the series converges to $f(x)$ on $0 < x < 2$. Thus, any extension will give us a useful series approximation to $f(x)$. We can therefore be clever, & choose the extension that means the least amount of work. (We can also try to choose an extension that yields a "continuous" fn.).

Recall: When solving PDE problems using sep'n of variables, we often ended up with solutions like

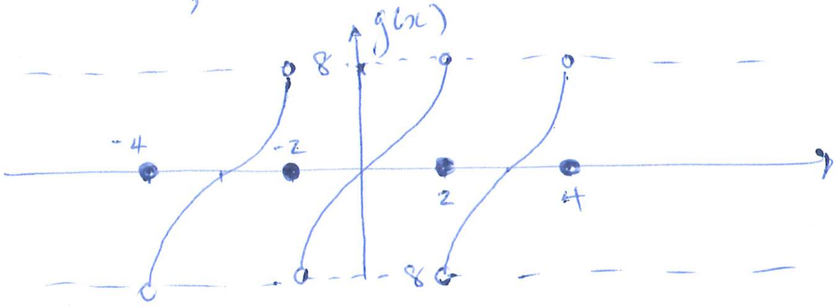
$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \qquad \text{(p575, text)}$$

So we need to be able to choose the appropriate extension so that we end up w/ the appropriate series (sine or cosine).

Convergence – Some Review (Rlhle badly done):

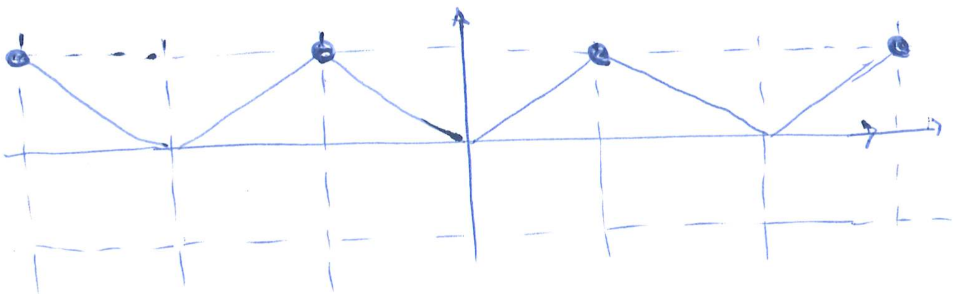
To what function g(x) do the Fourier series of the following functions f(x) converge? Is convergence uniform or pointwise?

a) $f(x) = x^3, \quad -2 < x < 2$



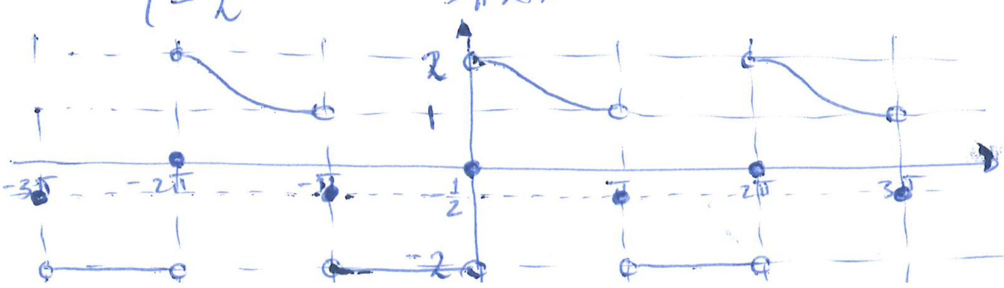
pointwise

b) $f(x) = |x|, \quad -1 < x < 1$



uniform

c) $f(x) = \begin{cases} \cos(x)+1 & 0 < x < \pi \\ -2 & -\pi < x < 0 \end{cases}$



FULL PROBLEM – START TO FINISH

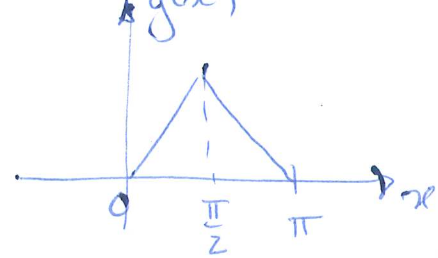
Example

Solve the IBVP:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} & \text{on } 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0 \\ u(x,0) = 0 \ \& \ \dfrac{\partial u}{\partial t}(x,0) = g(x) & \text{for } 0 < x < \pi \end{cases}$$

where

$$g(x) = -\left| x - \frac{\pi}{2} \right| + \frac{\pi}{2}$$



Step 1 – Separate

$$X''T = T''X \iff \frac{X''}{X} = \frac{T''}{T} = -\lambda$$

Step 2 – ODEs

(A) $X'' + \lambda X = 0, \quad X(0) = X(\pi) = 0$

(B) $T'' + \lambda T = 0$

Step 3 – Solve ODEs

(A) $\lambda < 0$ & $\lambda = 0 \Rightarrow$ only trivial sol'ns

$\lambda > 0$, let $\lambda = \omega^2$, then $X(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$

Apply $\begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases} \iff \begin{cases} c_1 = 0 \\ \omega = n,\ n \in \mathbb{Z} \end{cases}$

$\therefore\ \lambda = n^2$ and $X_n(x) = c_n \sin(nx)$

(B) $T(t) = \bar{a}_n \cos(nt) + \bar{b}_n \sin(nt)$

step 4 - Superposition

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)\left(\bar{a}_n \cos(nt) + \bar{b}_n \sin(nt)\right)$$

$$= \sum_{n=1}^{\infty} \sin(nx)\left(\tilde{a}_n \cos(nt) + \tilde{b}_n \sin(nt)\right)$$

step 5 - Apply ICs

$$u(x,0) = 0 \iff \sum_{n=1}^{\infty} d_n \sin(nx) = 0 \iff d_n = 0 \quad \forall n \in \mathbb{Z}$$

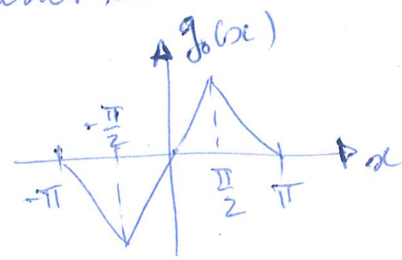
$$\frac{\partial u}{\partial t}(x,0) = g(x) \iff \sum_{n=1}^{\infty} n\tilde{b}_n \cos(nt)\Big|_{t=0} \sin(nx) = g(x)$$

$$\iff g(x) = \sum_{n=1}^{\infty} n\tilde{b}_n \sin(nx)$$

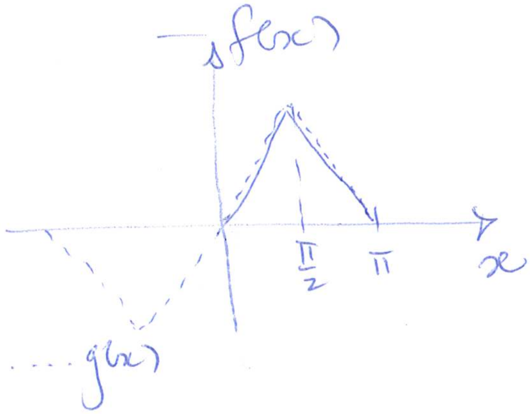
Fourier sine series coeffiecients b_n

step 6 - Calculate the Fourier coeffs

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{g_0(x)}_{\text{odd}}\ \underbrace{\sin(nx)}_{\text{odd}}}_{\text{even}}\,dx \quad \text{where}$$



$$= \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)\,dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx + \int_{\frac{\pi}{2}}^{\pi}(-x+\pi)\sin(nx)\,dx\right]$$



$$f(x) = \sum b_n \sin(nx)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{g(x)}_{\text{odd}}\ \cancel{dx}\ \underbrace{\sin(nx)}_{\text{odd}}}_{\text{even}}\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx + \int_{\frac{\pi}{2}}^{\pi} (-x+\pi)\sin(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{1}{n^2}\sin(nx) - \frac{x}{n}\cos(nx)\right)\Bigg|_0^{\frac{\pi}{2}} - \left(\frac{1}{n^2}\sin(nx) - \frac{x}{n}\cos(nx)\right)\Bigg|_{\frac{\pi}{2}}^{\pi} - \frac{\pi}{n}\cos(nx)\Bigg|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)\right) - \left(\frac{1}{n^2}\cancelto{0}{\sin(n\pi)} - \frac{\pi}{n}\cos(n\pi) - \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) + \frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)\right) - \frac{\pi}{n}\cos(n\pi) + \frac{\pi}{n}\cos\left(\frac{n\pi}{2}\right)\right]$$

6.9

$$b_n = \frac{2}{\pi}\left[\sin\left(\frac{n\pi}{2}\right)\left(\frac{1}{n^2}+\frac{1}{n^2}\right) + \cos\left(\frac{n\pi}{2}\right)\cancelto{0}{\left(\frac{-\pi}{2n}-\frac{\pi}{2n}+\frac{\pi}{n}\right)} + \cos(n\pi)\cancelto{0}{\left(\frac{+\pi}{n}-\frac{\pi}{n}\right)}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right)\right] = \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)$$

n even, let $n=2m$, m odd, then

$$b_n = b_{2m} = \frac{4}{(2m)^2\pi}\sin(m\pi) = 0$$

n odd, let $n = 2m+1$, then

$$b_{(2m+1)} = \frac{4}{(2m+1)^2\pi}\sin\left((2m+1)\frac{\pi}{2}\right)$$

$$= \frac{4}{(2m+1)^2\pi}(-1)^m \qquad m = 0, 1, 2, \ldots$$

$$g(x) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)^2}\frac{1}{\pi}(-1)^m \sin\left((2m+1)x\right)$$

$$\cancel{g(x) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)^2\pi}(-1)^m \sin\left((2m+1)x\right)}$$

✓

(correct

From before, when we applied the nonhom. IC., we have

$$g(x) = \sum_{n=1}^{\infty} n\tilde{b}_n \sin(nx) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

So

$$n\tilde{b}_n = b_n \iff (2m+1)\tilde{b}_{(2m+1)} = b_{(2m+1)}$$

$$\iff \tilde{b}_{(2m+1)} = \frac{1}{(2m+1)} b_{(2m+1)}$$

So

$$u(x,t) = \sum_{n=1}^{\infty} \tilde{b}_n \sin(nx)\sin(nt)$$

$$= \sum_{m=0}^{\infty} \tilde{b}_{2m+1} \sin((2m+1)x) \sin((2m+1)t)$$

$$= \sum_{m=0}^{\infty} \frac{1}{2m+1} b_{2m+1} \sin((2m+1)x) \sin((2m+1)t)$$

$$= \sum_{m=0}^{\infty} \frac{1}{2m+1} \frac{4}{(2m+1)^2} \frac{(-1)^m}{\pi} \sin((2m+1)x) \sin((2n+1)t)$$

$$= \sum_{m=0}^{\infty} \frac{4(-1)^m}{\pi(2m+1)^3} \sin((2m+1)x) \sin((2m+1)t)$$

If $f(x)$ is odd, then $f^2(x)$ is even.

Proof:

odd fn $f(x) = -f(-x)$

even fn $g(x) = g(-x)$

odd fn × odd fn $= f(x) \cdot f(x) = h(x)$

$$h(-x) = [-f(-x)][-f(-x)] = f(-x)f(-x)$$

$$= f^2(-x)$$

$$= (-f(x))(-f(x))$$

$$= f^2(x)$$

$$= h(x)$$

$\boxtimes$