More Heat Eqn problems

Prob'm

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \; t > 0$$

$$\text{BCs}: \; u(0,t) = 0, \quad u(\pi,t) + \frac{\partial u}{\partial x}(\pi,t) = 0 \qquad t > 0$$

Robin BCs

$$\text{IC}: \; u(x,0) = f(x) \qquad 0 < x < \pi$$

Sol'n

Step 1 Let $u(x,t) = F(x)\,G(t)$. Then

PDE: $FG' = F''G$ $\Leftrightarrow$ $\dfrac{G'}{G} = \dfrac{F''}{F} = -\lambda$

BCs: $\begin{cases} F(0)\,G(t) = 0 \\ F(\pi)\,G(t) + F'(\pi)\,G(t) = 0 \end{cases}$

For nontrivial sol'ns

$\begin{cases} F(0) = 0 \\ F(\pi) + F'(\pi) = 0 \end{cases}$

IC: $F(x)\,G(0) = f(x)$ not separable

Step 2 BVP: $\begin{cases} F'' + \lambda F = 0 \\ F(0) = 0, \; F(\pi) + F'(\pi) = 0 \end{cases}$ Ⓐ

t: $G' + \lambda G = 0$ Ⓑ

step3 So we have the BVP

Ⓐ
$$\begin{cases} F''+\lambda F=0 \\ F(0)=0 \\ F(\pi)+F'(\pi)=0 \end{cases}$$

Solving the ODE we find

<u>Case 1: $\lambda<0$</u>, $\lambda=-\mu^2$

$$F(x)=c_1\cosh(\mu x)+c_2\sinh(\mu x)$$

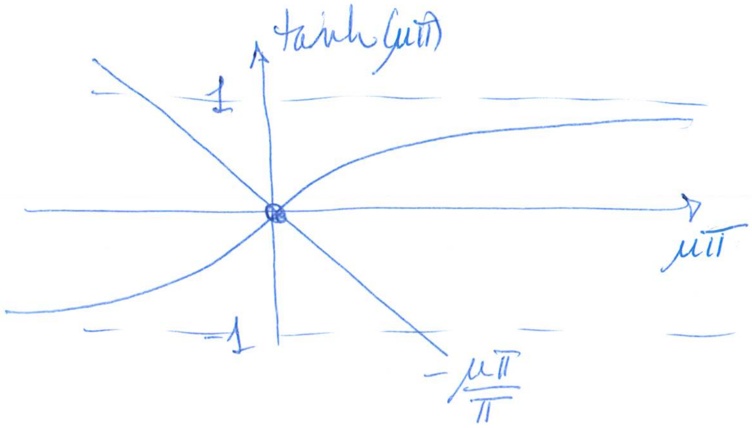
$$F(0)=0 \iff c_1=0$$

$$F(\pi)+F'(\pi)=0 \iff c_2\sinh(\mu\pi)+c_2\mu\cosh(\mu\pi)=0$$

$$\iff c_2=0 \text{ or } \sinh(\mu\pi)+\mu\cosh(\mu\pi)=0$$

$$\frac{\sinh(\mu\pi)}{\cosh(\mu\pi)}=-\frac{\mu\pi\cosh(\mu\pi)}{\pi\cosh(\mu\pi)}$$

$$\frac{\sinh}{\cosh}=\frac{e^x+\cancel{e^{-x}}}{e^x-\cancel{e^{-x}}}=1 \text{ as } x\to\infty$$



Only trivial solutions are possible.

Case 2: $\lambda = 0$

$F(x) = Ax + B$

$F(0) = 0 \iff B = 0$

$F(\pi) + F'(\pi) = 0 \iff A \cdot \pi + A = 0 \iff A(\pi + 1) = 0 \iff A = 0$

Only trivial sol'ns are possible.

Case 3: $\lambda > 0$, $\lambda = +\mu^2$

$F(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$

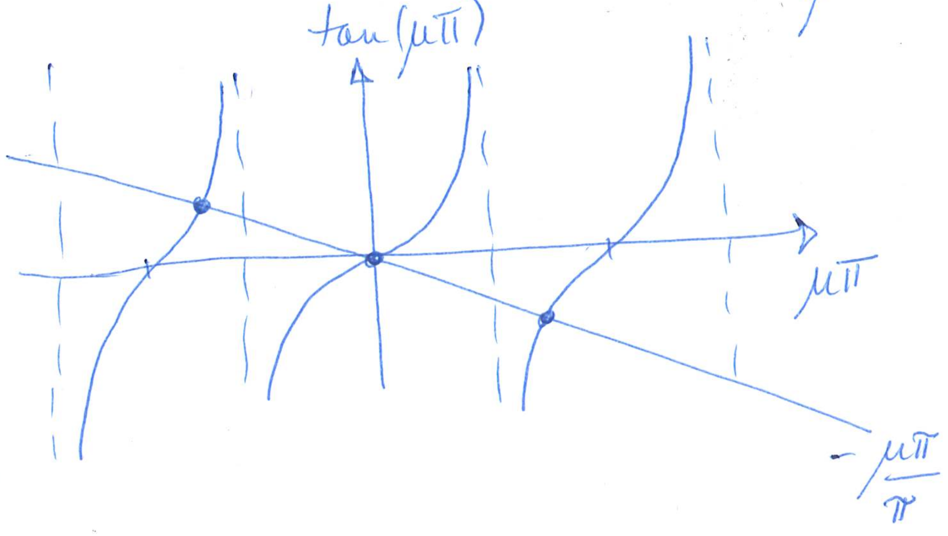
$F(0) = 0 \iff c_1 = 0$

$F(\pi) + F'(\pi) = 0 \iff c_2 \sin(\mu\pi) + c_2 \mu \cos(\mu\pi) = 0$

$\iff c_2 = 0$ or

$\sin(\mu\pi) + \mu\cos(\mu\pi) = 0$

$\Rightarrow \tan(\mu\pi) = -\frac{\mu\pi}{\pi}$



There are multiple intersections.

So $\mu_n = -\tanh(\mu_n \pi)$ (transcendental eqn) → eigenvals.

∴ $F_n(x) = B_n \sin(\mu_n x)$ → eigenfns

Ⓑ The time-dependent problem

$$G_T' + \lambda G_T = 0 \Leftrightarrow G_T' + \mu^2 G_T = 0 \Leftrightarrow G_T' = -\mu^2 G_T$$

$$\Leftrightarrow G_{T_n}(t) = c_n e^{-\mu_n^2 t}$$

and so we have

$\boxed{\text{Step 4+5}}$

$$u(x,t) = F(x)\,G(t)$$

$$= \sum_{n=1}^{\infty} B_n e^{-\mu_n^2 t} \sin(\mu_n x)$$

where $B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(\mu_n x)\,dx$

What to we do when the PDE or the BCs are not homogeneous? We assume that

$$u(x,t) = \underset{\substack{\uparrow \\ \text{steady-} \\ \text{state}}}{v(x)} + \underset{\substack{\uparrow \\ \text{transient}}}{w(x,t)}$$

Recall, ODEs:

$$my'' + by' + ky = f(t)$$

then

$$y(t) = \underset{\substack{\downarrow \\ \text{transient} \\ \text{(non-transient} \\ \text{case: resonance)}}}{\boxed{\text{homogeneous solution}}} + \underset{\substack{\downarrow \\ \text{steady state}}}{\boxed{\text{particular solution}}}$$

Ex 2

$$\begin{array}{l} \\ \text{BCs:} \\ \text{IC:} \end{array}\begin{cases} \dfrac{\partial u}{\partial t} = \beta \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L, \quad t > 0 \\ u(0,t) = U_1, \quad u(L,t) = U_2, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

Let $u(x,t) = v(x) + w(x,t)$.

Then

$$\frac{\partial u}{\partial t} = \frac{\partial w}{\partial t}, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{d^2 v}{dx^2} + \frac{\partial^2 w}{\partial x^2}$$

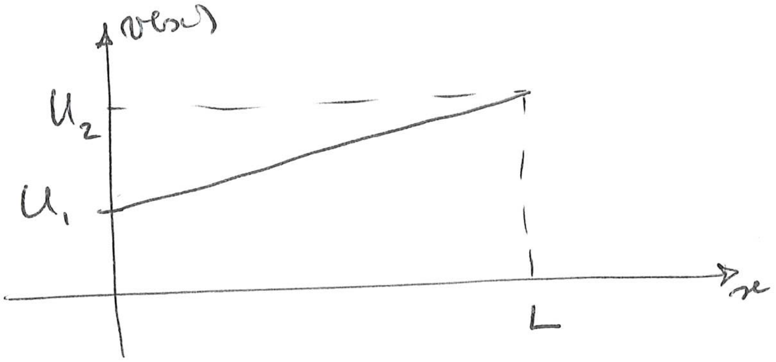
and the PDE problem becomes

$$\begin{cases} \dfrac{\partial w}{\partial t} = \beta\left(\dfrac{d^2 v}{dx^2} + \dfrac{\partial^2 w}{\partial x^2}\right), & 0 < x < L, \quad t > 0 \\ v(0) + w(0,t) = U_1, \quad v(L) + w(L,t) = U_2 \\ v(x) + w(x,0) = f(x), & 0 < x < L \end{cases}$$

Now, let $t \to \infty$. Then, ∵ $w(x,t)$ is transient (by assumpØ), we have

$$\begin{cases} \beta v'' = 0, & 0 < x < L \\ v(0) = U_1, \quad v(L) = U_2 \end{cases} \Leftrightarrow \boxed{v(x) = \frac{(U_2 - U_1)x}{L} + U_1}$$

steady-state sol'n



Then, we have remaining

$$\begin{cases} \dfrac{\partial w}{\partial t} = \beta \dfrac{\partial^2 w}{\partial x^2}, & 0 < x < L, \quad t > 0 \\ w(0,t) = w(L,t) = 0 & \text{Dirichlet} \\ w(x,0) = f(x) - u_1 - \dfrac{(u_2 - u_1)}{L} x \end{cases}$$

We know how to solve this! The formal sol'n is

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

↳ Fourier sine series

where

$$c_n = \frac{2}{L} \int_0^L \left(f(x) - u_1 - \frac{(u_2 - u_1)}{L} x \right) \sin\left(\frac{n\pi x}{L}\right) dx$$

Recap - PDE + BCs

I if homogeneous → Apply Sep'n of Vars

→ solve the BVP

obtain eigenvalues + eigenfunctions

→ form the full solution

(superposition + Fourier Series)

→ find the coefficients

(Euler Formulas)

II if non-homogeneous

→ Let $u(x,t) = v(x) + w(x,t)$

$v(x)$ ↑ steady-state

$w(x,t)$ ↑ transient

→ separate into a steady-state problem $(t \to \infty)$ + a homogeneous problem

→ solve the ss problem

→ go to I

→ put both sol'ns together [illegible]