Laplace's Equation (§10.7)

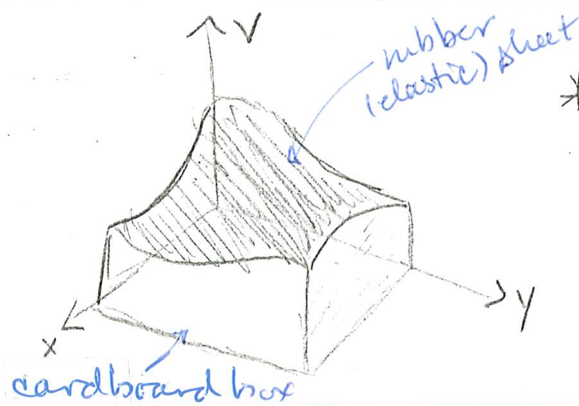
Rectangular Coordinates

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{OR} \quad \nabla^2 u = \Delta u = 0$$

$\frac{\partial u}{\partial t} = 0 \Rightarrow$ steady-state sol'ns

Applications?
- level curves of stream flow
- electrostatic & gravitational potentials
- pattern formation.

Physical example

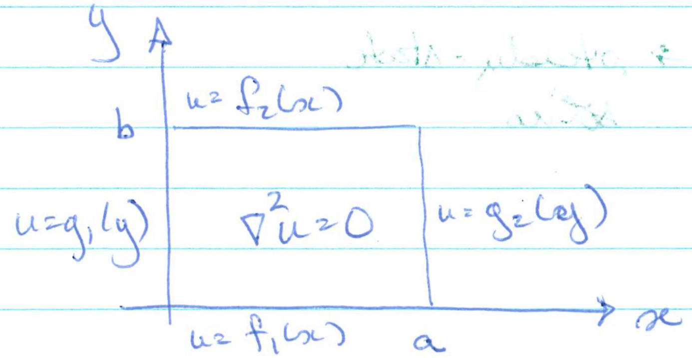


*The height $V(x,y)$ of the Sheet above the point (x,y) will satisfy Laplace's equation.

9b-2

General Dirichlet Problem on a Rectangle

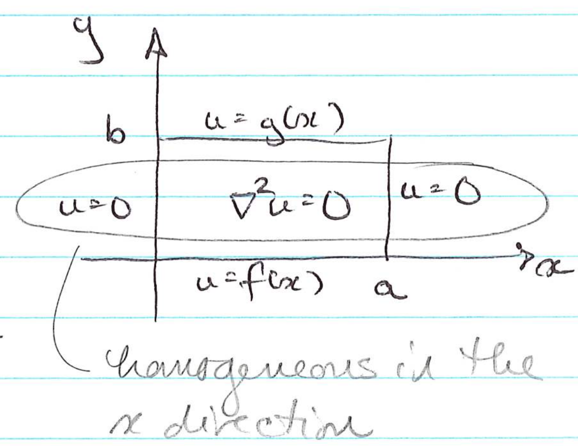
$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & 0<x<a,\ 0<y<b \\ u(x,0) = f_1(x),\quad u(x,b) = f_2(x) & 0<x<a \\ u(0,y) = g_1(y),\quad u(a,y) = g_2(y) & 0<y<b \end{cases}$$



* To apply separation of vars, require homogeneous BCs in at least one direction.

Ex 1

$$(1)\ldots \begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & \\ u(0,y) = u(a,y) = 0 & 0<y<b \\ u(x,0) = f(x) & \\ u(x,b) = g(x) & \Big\} \ 0<x<a \end{cases}$$



Sol'n - step 1

Let $u(x,y) = F(x)G(y)$. Then (1) becomes

$$F''G + FG'' = 0 \iff \frac{F''}{F} + \frac{G''}{G} = 0$$

* Which do we solve 1st? $\because$ the F problem is

homogeneous, we write

$$\frac{F''}{F} = -\frac{G''}{G} = -\lambda$$

step 2

$$\text{(A)} \begin{cases} F'' + \lambda F = 0 \\ F(0) = F(a) = 0 \end{cases} \qquad \text{(B)}\ G'' - \lambda G = 0$$

step 3

solve (A). We have done this many times! The nontrivial sol'ns are obtained when $\lambda > 0$. Let $\lambda = \omega^2$, then the solutions are

$$F_n(x) = c_n \sin\left(\frac{n\pi x}{a}\right), \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2, \ n \in \mathbb{N} \quad \text{- - - - - (2)}$$

solve (B).

$$G'' - \left(\frac{n\pi}{a}\right)^2 G = 0 \iff G_n(y) = \tilde{d}_{1n} e^{-\left(\frac{n\pi}{a}\right)y} + \tilde{d}_{2n} e^{\left(\frac{n\pi}{a}\right)y}$$

$$\text{OR} \quad = d_{1n}\cosh\left(\frac{n\pi y}{a}\right) + d_{2n}\sinh\left(\frac{n\pi y}{a}\right)$$

* $\boxed{\text{TRICK**}}$ We choose $G(y)$ so that only one coefficient survives at each boundary when we apply the BCs in y.

How do we do this?

9b-2f

BCs: $\begin{cases} u(x,0) = f(x) \\ u(x,b) = g(x) \end{cases} \Leftrightarrow \begin{cases} F(x)\,G(0) = f(x) \\ F(x)\,G(b) = g(x) \end{cases}$

$\therefore$ @ $G(0)$ we'd like either the d_{1n} or d_{2n} term to dropout, and at $G(b)$ we'd like the other term to drop out.

So we choose

$$G_n(y) = d_{1n}\sinh\left(\frac{n\pi(b-y)}{a}\right) + d_{2n}\sinh\left(\frac{n\pi y}{a}\right) \quad \ldots (3)$$

Check:

$$\begin{cases} G_n(0) = d_{1n}\sinh\left(\frac{n\pi b}{a}\right) & \to d_{2n} \text{ absent} \\ G_n(b) = d_{2n}\sinh\left(\frac{n\pi b}{a}\right) & \to d_{1n} \text{ absent} \end{cases}$$

* //

Step 4: Superposition

$$u(x,y) = F(x)\,G(y)$$
$$= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)\left[d_{1n}\sinh\left(\frac{n\pi(b-y)}{a}\right) + d_{2n}\sinh\left(\frac{n\pi y}{a}\right)\right] \quad \ldots (4)$$

Step 5: Apply 2^{nd} set of BCs

$$\begin{cases} u(x,0) = f(x) \\ u(x,b) = g(x) \end{cases} \Rightarrow \begin{cases} f(x) = \sum_{n=1}^{\infty} \underbrace{c_n d_{1n}}_{A_n} \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \\ g(x) = \sum_{n=1}^{\infty} \underbrace{c_n d_{2n}}_{B_n} \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \end{cases}$$

$$\therefore \begin{cases} A_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx \\ B_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a g(x) \sin\left(\frac{n\pi x}{a}\right) dx \end{cases} \quad \dots (5)$$

and

$$u(x,y) = \sum_{n=1}^{\infty} \left[A_n \sinh\left(\frac{n\pi (b-y)}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right) \right] \sin\left(\frac{n\pi x}{a}\right) \quad \dots (6)$$

Equations (6) & (5) give the solution to (1).

Ex 2

How would you define $G_n(y)$ for (1) if the BCs wrt y were changed to

$$u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial y}(x,b) = g(x)?$$

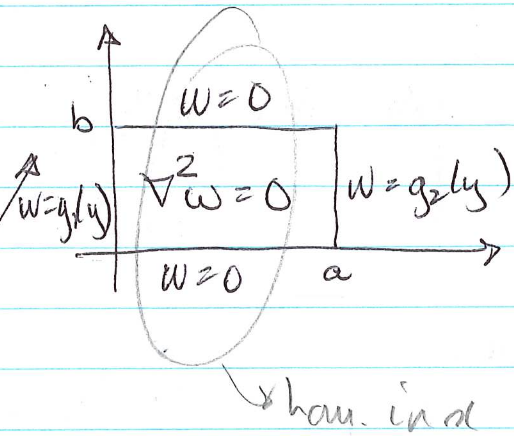
Ans:

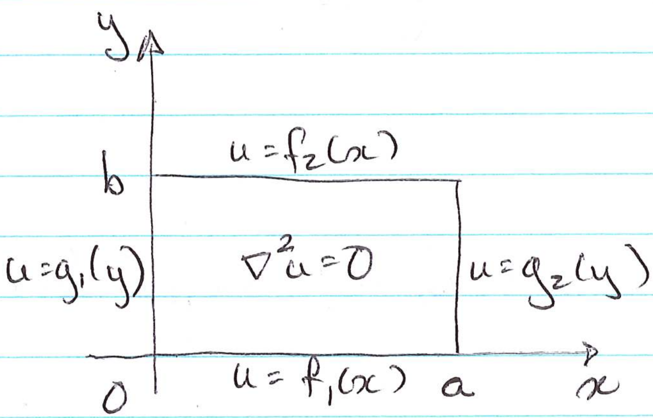
$$G_n(y) = d_{1n} \sinh\left(\frac{n\pi y}{a}\right) + d_{2n} \cosh\left(\frac{n\pi (b-y)}{a}\right)$$

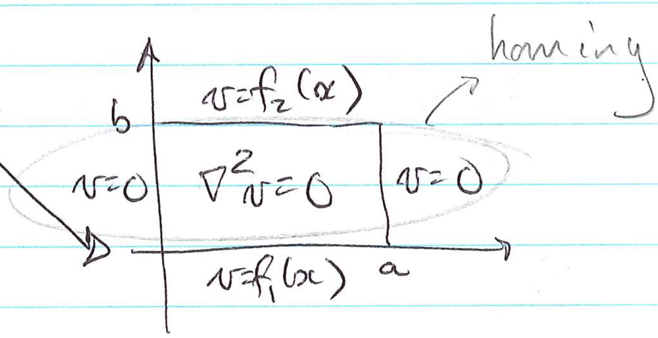
Ex 3

General case:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & 0 < x < a \;\&\; 0 < y < b \\ u(x,0) = f_1(x), \quad u(x,b) = f_2(x) & 0 < x < a \\ u(0,y) = g_1(y), \quad u(a,y) = g_2(y) & 0 < y < b \end{cases}$$







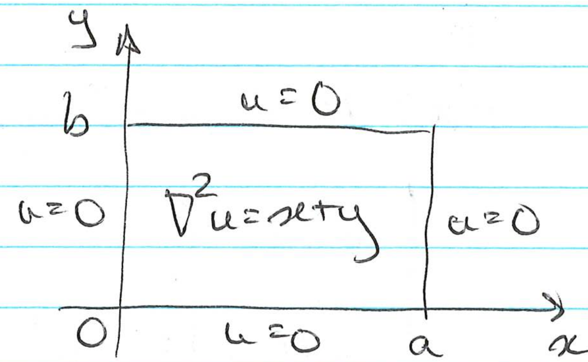
Let $u(x,y) = w(x,y) + v(x,y)$, where (see sketch). Then

$$\nabla^2 u = 0 \iff \nabla^2 w + \nabla^2 v = 0$$

Assume $\begin{cases} \nabla^2 w = 0 \\ \nabla^2 v = 0 \end{cases}$. Then we have two Laplace problems to consider (see Ex 1 for ea one).

Ex 4 - Nonhomogeneous Laplace Eqn

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = x+y & 0<x<a,\ 0<y<b \\ u(0,y) = u(a,y) = 0 \\ u(x,0) = u(x,b) = 0 \end{cases}$$



Soln

Assume * $u(x,y) = \underbrace{u_p(x,y)}_{\substack{\text{particular} \\ \text{soln} \\ \nabla^2 u_p = x+y}} + \underbrace{w(x,y)}_{\substack{\text{hom.} \\ \text{soln, i.e. } \nabla^2 w = 0}}$

Then

$$\nabla^2 u = \nabla^2 w + \nabla^2 u_p = 0 + x+y$$

Now assume * $u_p(x,y) = u_1(x) + u_2(y)$.

Then

$$\nabla^2 u_p = \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = x+y$$

and set

$$\frac{\partial^2 u_1}{\partial x^2} = x \qquad\qquad \frac{\partial^2 u_2}{\partial y^2} = y$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$u_1 = \frac{x^3}{6} + c_1 x + d_1 \qquad\qquad u_2 = \frac{y^3}{6} + c_2 y + d_2$$

arbitrary! So set to 0. :)

So a possible solution is

$$u_p(x,y) = \frac{x^3}{6} + \frac{y^3}{6}$$

Now do w-problem; $w = u - u_p$

$$\begin{cases} \nabla^2 w = 0 & 0 < x < a, \; 0 < y < b \\ w(0,y) = -\frac{y^3}{6}, \quad w(a,y) = -\frac{a^3}{6} - \frac{y^3}{6} & 0 < y < b \\ w(x,0) = -\frac{x^3}{6}, \quad w(x,b) = -\frac{x^3}{6} - \frac{b^3}{6} & 0 < x < a \end{cases}$$

Now we have a problem just like Ex 3.