Laplace Eqn: Polar Coords

disc semi-circle wedge annulus

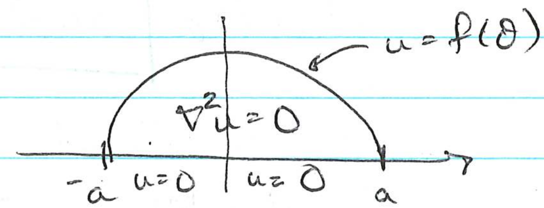
$u(x,y) = u(r,\theta)$ $\quad \begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$

With some work (chain rule fun) we find

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Ex. 1

$$(7) \ldots \begin{cases} \nabla^2 u = 0 & 0 < r < a,\ 0 < \theta < \pi \\ u(r,0) = u(r,\pi) = 0 & 0 < r < a \\ u(a,\theta) = f(\theta) & 0 < \theta < \pi \end{cases}$$



Soln

Step 1: Separate

$u(r,\theta) = R(r)\,T(\theta)$ and (7a) becomes

$$R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' = 0 \quad (20)$$

Qc-2

$1) \Leftrightarrow \frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{T''}{T}$

We have homogeneous BCs wrt θ, so we write

$$\frac{T''}{T} = -\frac{r^2R''}{R} - \frac{rR'}{R} = -\lambda$$

Step 2: ODEs

Ⓐ $\begin{cases} T'' + \lambda T = 0 \\ T(0) = T(\pi) = 0 \end{cases}$ Ⓑ $r^2R'' + rR' - \lambda R = 0$

Step 3: solve

Ⓐ For nontrivial solutions, $\lambda > 0$, set $\lambda = \omega^2$.
Then

$$T_n(\theta) = C_n \sin(n\theta), \quad n \in \mathbb{N}$$

Ⓑ $r^2R'' + rR' - n^2R = 0$

This is a $\boxed{\text{Cauchy-Euler equation.}}$ To solve, we set $R(r) = r^p$. Then we obtain the characteristic equation

$$p(p-1) + p - n^2 = 0 \Leftrightarrow p^2 - n^2 = 0$$
$$\Leftrightarrow p = \pm n$$

$\therefore R(r) = d_{1n} r^n + d_{2n} r^{-n}$

$\because$ The domain includes points $r \to 0$; we must have $d_{2n} = 0$, & so

$$R(r) = d_n r^n$$

step 4: Superposition

$$u(r,\theta) = \sum_{n=1}^{\infty} c_n \sin(n\theta)\, d_n r^n$$

$$= \sum_{n=1}^{\infty} b_n r^n \sin(n\theta) \quad \dots\dots (7)$$

step 5: Apply remaining BC

$$u(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} b_n a^n \sin(n\theta)$$

$$\therefore b_n a^n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin(n\theta)\, d\theta \quad \dots (8)$$

The formal solution is given by (7) + (8) together.

$\boxed{9c\text{-}4}$