Math 319, some code to compute Fourier Seres coefficients and then plot the Fourier Series

```
> with(plots) :
```

Consider the following piecewise continuous function defined on the interval 0<x<2:

```
> fB := x → piecewise(And(0 ≤ x, x < 1), 2·x, And(1 ≤ x, x < 2), 2, Or(x < 0, x ≥ 2), 0);
```

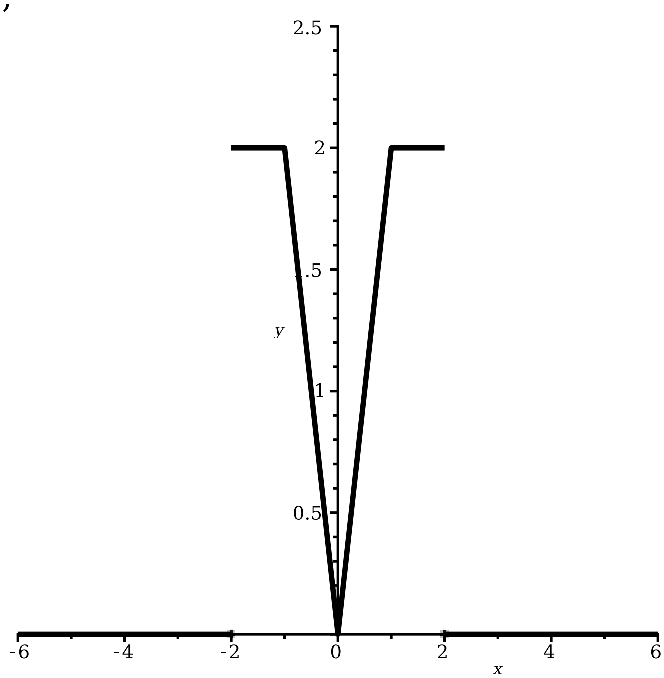
$$fB := x \mapsto \begin{cases} 2 \cdot x & 0 \le x \wedge x < 1 \\ 2 & 1 \le x \wedge x < 2 \\ 0 & x < 0 \vee 2 \le x \end{cases} \quad (1)$$

To find the Fourier Series representation of this function, we need to extend it. There are several options.

Fourier Cosine Series

We can plot part of the even extension of fB(x):

```
> plot([fB(x) + fB(-x)], x = -6..6, y = 0..2.5, discont = true, colour = black, thickness = 2);
```



The Fourier Series approximation of the even extension of fB(x) is the Fourier Cosine Series. We compute the coefficients:

```
> a0 := simplify(expand(int(2·x, x = 0..1) + int(2, x = 1..2)));
```

$$a0 := 3 \quad (2)$$

```
> an := simplify(expand(int(2·x·cos(n·Pi·x/2), x = 0..1) + int(2·cos(n·Pi·x/2), x = 1..2)));
```

$$an := \frac{8\,n\,\pi \sin\left(\frac{n\,\pi}{2}\right)\cos\left(\frac{n\,\pi}{2}\right) + 8\cos\left(\frac{n\,\pi}{2}\right) - 8}{n^2\,\pi^2} \quad \mathbf{(3)}$$

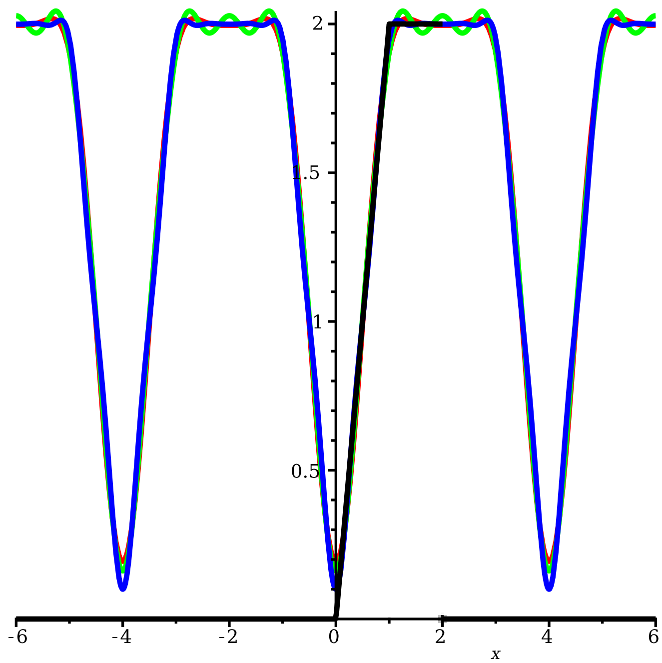
> $fc := (x, nmax) \rightarrow \frac{3}{2} + add\left(\left(\frac{8}{(n\cdot \mathrm{Pi})^2}\cdot\left(\cos\left(\frac{n\cdot\mathrm{Pi}}{2}\right) - 1\right)\right)\cdot\cos\left(\frac{n\cdot\mathrm{Pi}\cdot x}{2}\right), n = 1..nmax\right);$

Warning, (in fc) `n` is implicitly declared local

$$fc := (x, nmax) \mapsto \frac{3}{2} + add\left(\frac{8\cdot\left(\cos\left(\frac{n\cdot\pi}{2}\right) - 1\right)\cdot\cos\left(\frac{n\cdot\pi\cdot x}{2}\right)}{n^2\cdot\pi^2}, n = 1..nmax\right) \quad \mathbf{(4)}$$

>

> $plot([fc(x, 3), fc(x, 5), fc(x, 7), fB(x)], x = -6..6, colour = [red, green, blue, black], thickness = 2, discont = true);$



We see that the fourier series is converging to the even extension of fB(x), though with lots of ringing.

Fourier Sine Series

Computing the coefficients

> $bn := expand\left(int\left(2\cdot x\cdot\sin\left(\frac{n\cdot\mathrm{Pi}\cdot x}{2}\right), x = 0..1\right) + int\left(2\cdot\sin\left(\frac{n\cdot\mathrm{Pi}\cdot x}{2}\right), x = 1..2\right)\right);$

$$bn := \frac{8 \sin\left(\frac{n\pi}{2}\right)}{n^2 \pi^2} - \frac{4 \cos(n\pi)}{n\pi} \quad \textbf{(5)}$$

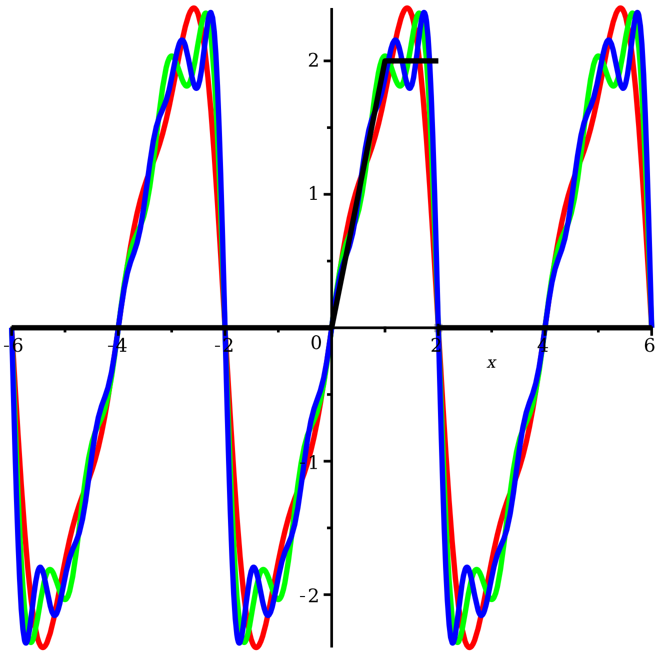
Using the fact that n is an integer, we can simplify the coefficients above, and obtain the fourier sine series:

$$> fs := (x, nmax) \rightarrow add\left(\left(\frac{8}{(n \cdot \mathrm{Pi})^2} \cdot \sin\left(\frac{n \cdot \mathrm{Pi}}{2}\right) - \frac{4}{n \cdot \mathrm{Pi}} \cdot (-1)^n\right) \cdot \sin\left(\frac{n \cdot \mathrm{Pi} \cdot x}{2}\right), n = 1 ..nmax\right);$$

Warning, (in fs) `n` is implicitly declared local

$$fs := (x, nmax) \mapsto add\left(\left(\frac{8 \cdot \sin\left(\frac{n \cdot \pi}{2}\right)}{n^2 \cdot \pi^2} - \frac{4 \cdot (-1)^n}{n \cdot \pi}\right) \cdot \sin\left(\frac{n \cdot \pi \cdot x}{2}\right), n = 1 ..nmax\right) \quad \textbf{(6)}$$

$$> plot([fs(x, 3), fs(x, 5), fs(x, 7), fB(x)], x = -6 ..6, colour = [red, green, blue, black], thickness = 2, discont = true);$$



We see that the fourier series is converging to the odd extension of fB(x), though with lots of ringing.

Other extensions are possible. These two, the even and odd extensions, are the ones that lend themselves most easily to a Fourier Series representation, as they are composed of just cosines (in the odd extension case) and just sines (in the even extension case). Other extensions would lead to a mix of cosine and sine functions in the series.