A#1 - Sol'ns

1. $f(x) = ax + b$

$f'(x) = a$ (does not depend on x)

$\therefore$ If a fixed point exists, then it is

- stable if $|a| < 1$ (attracting, sink)
- unstable if $|a| > 1$ (repelling, source)

(Aside: The fixed point is given by

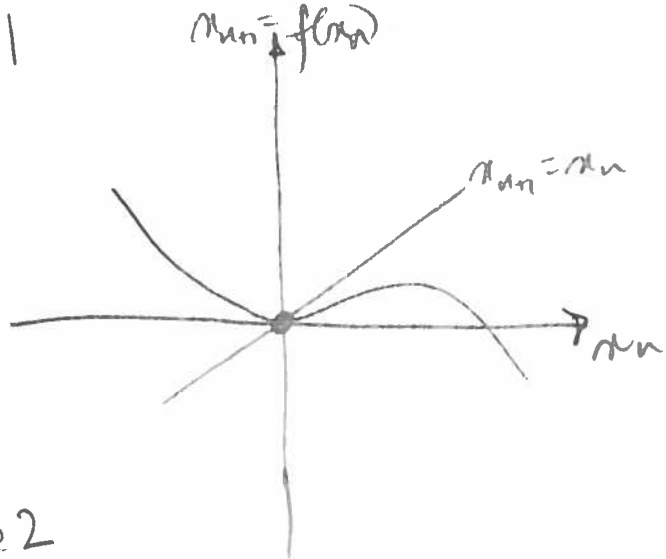
$$x^* = f(x^*) \iff x^* = ax^* + b$$

$$\iff x^* = \frac{b}{1-a}$$

And so a fixed point exists if $a \neq 1$.)

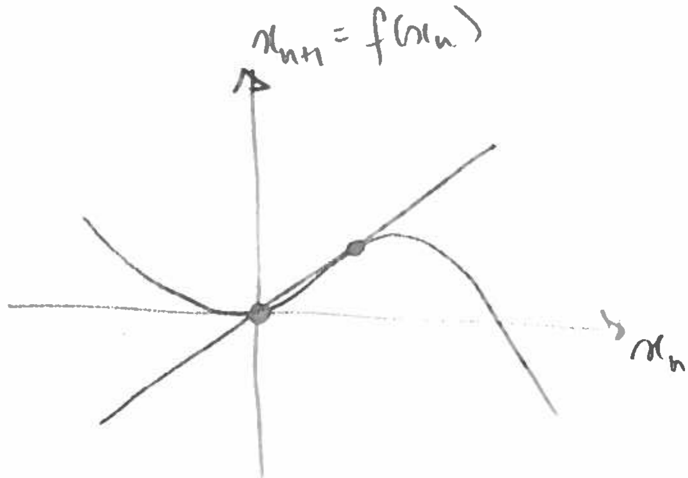
2a) $f(x) = a(x^2 - x^3) = ax^2(1-x)$

case 1

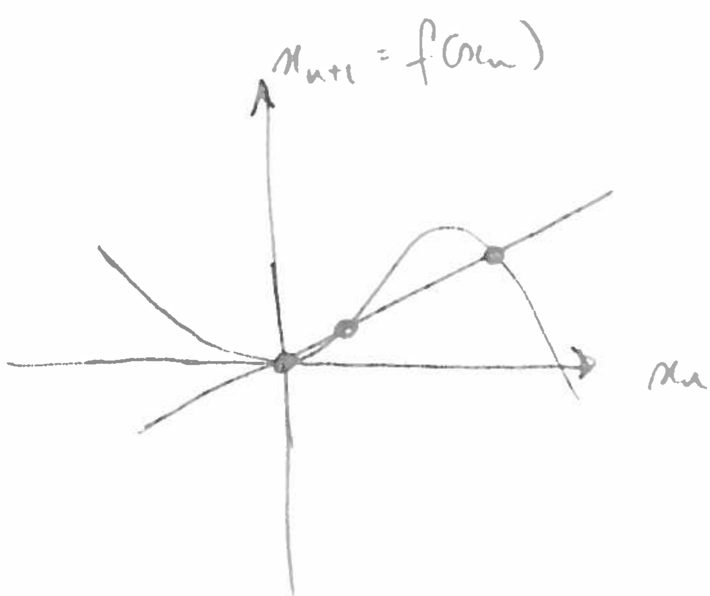


one steady-state

case 2



two steady-states



three steady-states

2) b) $x^* = a(x^{*2} - x^{*3}) \Leftrightarrow x^* = ax^{*2}(1-x^*) \Leftrightarrow$ //

// $\Leftrightarrow 1 = ax^*(1-x^*)$ or $x^* = 0$

$\Leftrightarrow ax^{*2} - ax^* + 1 = 0$

$\Leftrightarrow x^* = \dfrac{a \pm \sqrt{a^2 - 4a}}{2a}$ and 0

The $x^* = 0$ steady state exists $\forall a$.

The other two steady states exist when

$a^2 - 4a > 0 \Leftrightarrow a(a-4) > 0 \Leftrightarrow a > 4$.

c) Stability

$f'(x) = a(2x - 3x^2)$

$x^* = 0$

$f'(0) = 0$, $\therefore$ this steady state is always stable (Allee effect).

(d) The nonzero steady states at $a = \frac{9}{2}$

$x^* = \frac{1}{9}\left(\frac{9}{2} \pm \sqrt{\left(\frac{9}{2}\right)^2 - 4\left(\frac{9}{2}\right)}\right) = \frac{1}{2} \pm \frac{1}{9}\sqrt{\frac{81}{4} - 18}$

$= \frac{1}{2} \pm \frac{1}{18}\sqrt{81 - 72} = \frac{1}{2} \pm \frac{1}{18}\sqrt{9} = \frac{1}{2} \pm \frac{3}{18} = \frac{1}{2} \pm \frac{1}{6}$

$\therefore x^* = \frac{2}{6}$ or $\frac{4}{6}$ $= \frac{1}{3}$ or $\frac{2}{3}$

Stability:

$$f'\left(\frac{1}{3}\right) = \frac{9}{2}\left(2\left(\frac{1}{3}\right) - 3\left(\frac{1}{3}\right)^2\right) = \frac{9}{2}\left(\frac{2}{3} - \frac{1}{3}\right)$$

$$= \frac{9}{2}\left(\frac{1}{3}\right) = \frac{3}{2} > 1$$

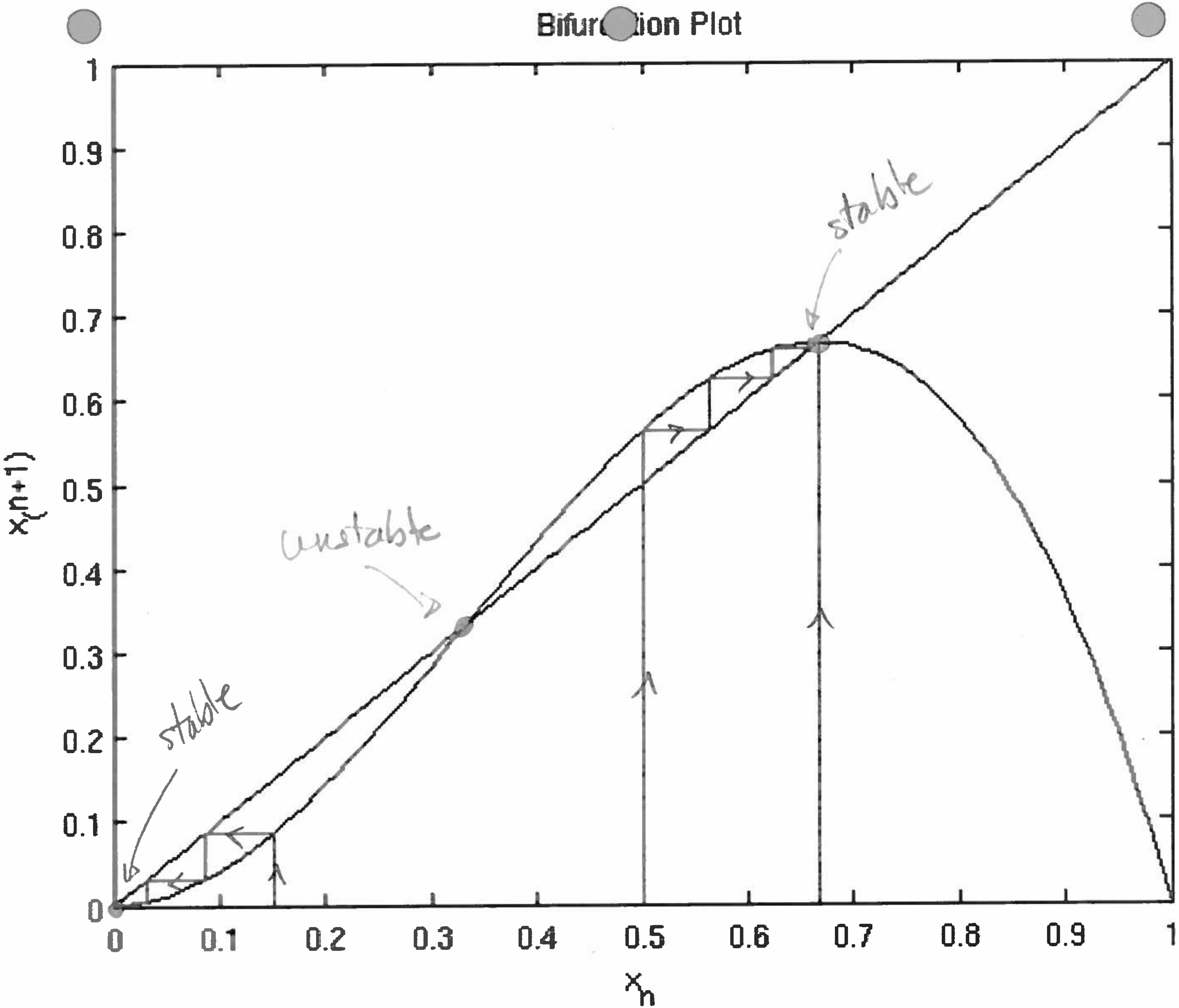
$\therefore$ this steady state is unstable.

$$f'\left(\frac{2}{3}\right) = \frac{9}{2}\left(2\left(\frac{2}{3}\right) - 3\left(\frac{2}{3}\right)^2\right) \cdot \frac{9}{2}\left(\frac{4}{3} - \frac{4}{3}\right)$$

$$= 0$$

$\therefore$ this steady state is stable.

e) Using software we find that cobwebbing confirms our analysis (see next p.)


Bifurc[ation] Plot
stable
unstable
stable
$x_{(n+1)}$
x_n
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

5

3(b)

From Fig 3.1 we see that as a increases the slope $|f'(x)|$ at each fixed point increases. From Fig 3.2 we see that $|f'(x^*)| \simeq 1$ when $a = 4$, and $|f'(x^*)| > 1$ when $a = 5$. So we expect the steady state near 0.5 to become unstable as a increases past 4. At this point, the system should exhibit new behaviour because both steady states of f will be unstable.

3(c)

The bifurcation plot on the next page shows that as a increases the system undergoes a series of period-doubling bifurcations leading to chaos.

Consistent with 3(b), as a increases past $a = 4$, the ~~system~~ single steady state gives way to a period-2 orbit.

$f(x) = 1 - ax^2(1-x)$

Map for a from 4 to 5

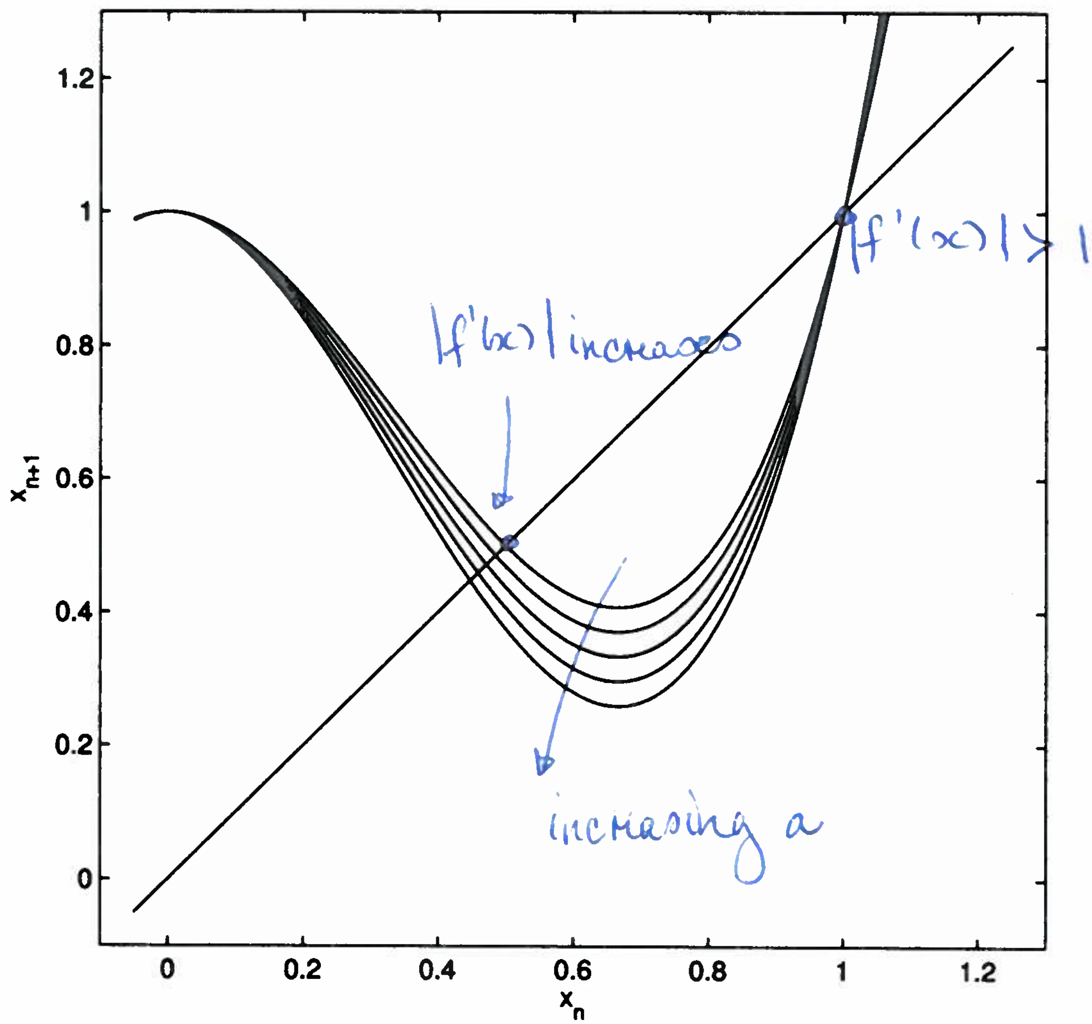


Fig 3.1

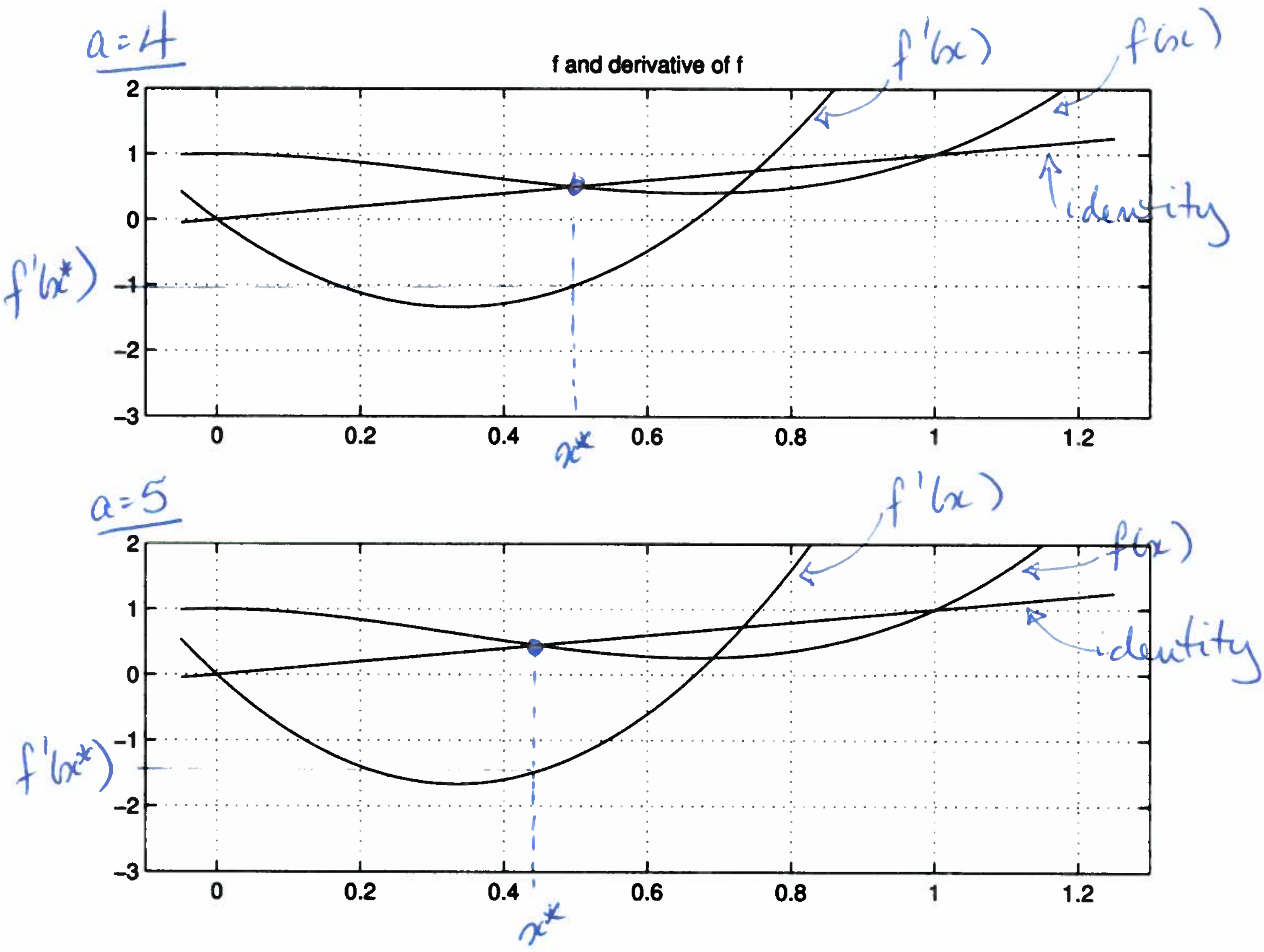
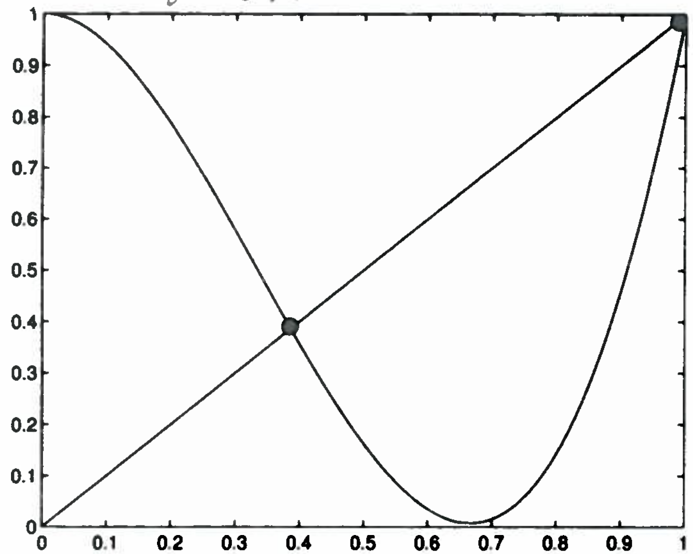

a=4
f and derivative of f
f'(x)
f(x)
identity
f'(x*)
x*
2
1
0
-1
-2
-3
0
0.2
0.4
0.6
0.8
1
1.2
a=5
f'(x)
f(x)
identity
f'(x*)
x*

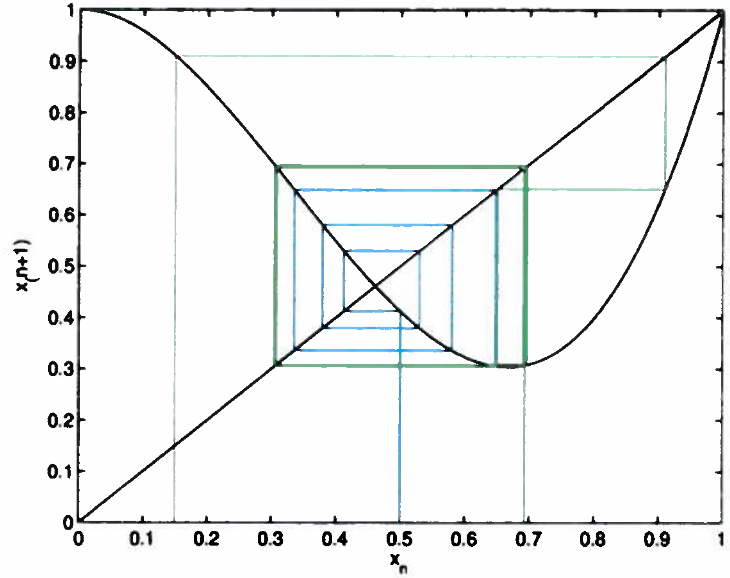
Fig 3.2

(9)

a = 6.7

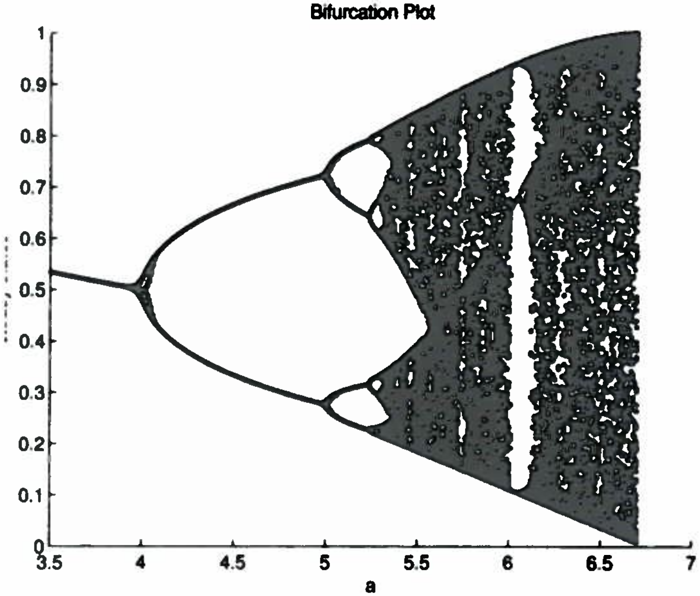


Cobweb Plot, a = 4.7

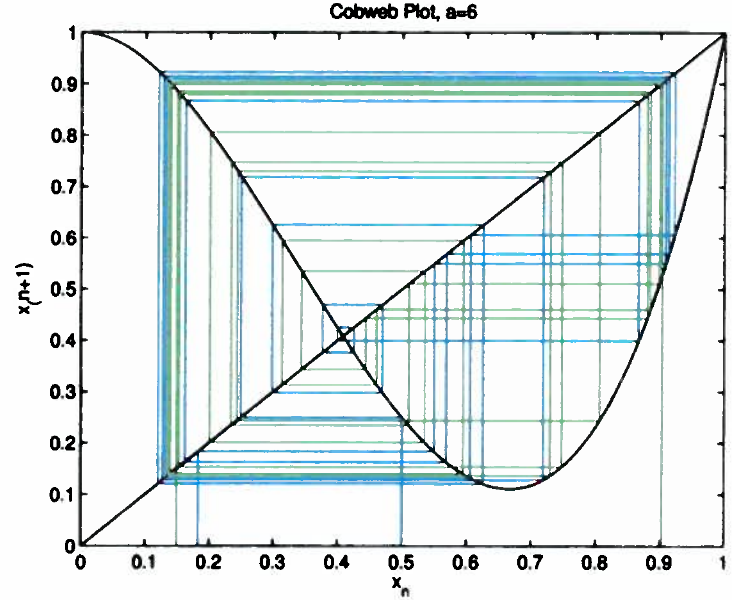


2-cycle

Bifurcation Plot



Cobweb Plot, a=6



chaos?

3)