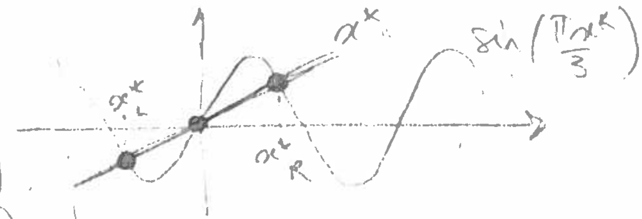
A#2

Sol'ns

1. a) $f(x,y) = \left(\sin\left(\frac{\pi x}{3}\right), \frac{y}{2}\right)$

fixed pts:

$$\begin{cases} x^* = \sin\left(\frac{\pi x^*}{3}\right) \\ y^* = \frac{y^*}{2} \end{cases} \Longleftrightarrow$$



$$y^* = 0$$

The fixed points are at

$$(0,0),\ (x^*_R, 0),\ (x^*_L, 0),$$

and $x^*_L = -x^*_R$, $x^*_R \simeq 2$ (from plot).

b) See Maple

2\. (i) (a) eigenvalues:

$$\begin{vmatrix} 2-\lambda & 0.5 \\ 2 & -0.5-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)\left(-\frac{1}{2}-\lambda\right) - \frac{1}{2}2 = 0$$

$$\Leftrightarrow -1 - 2\lambda + \frac{\lambda}{2} + \lambda^2 - 1 = 0$$

$$\Leftrightarrow \lambda^2 + \left(\frac{1}{2}-2\right)\lambda - 2 = 0$$

$$\Leftrightarrow \lambda^2 - \frac{3}{2}\lambda - 2 = 0$$

$$\Leftrightarrow 2\lambda^2 - 3\lambda - 4 = 0$$

$$\Leftrightarrow \lambda = \frac{3 \pm \sqrt{9+32}}{4}$$

$$= \frac{3 \pm \sqrt{41}}{4} = 2.3, -0.85$$

So the fixed point is a <u>saddle</u>.

(ii) (a) See Maple

(i)

(b) eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)^2 + 2 = 0$$

$$\Leftrightarrow \lambda^2 - 4\lambda + 4 + 2 = 0$$

$$\Leftrightarrow \lambda^2 - 4\lambda + 6 = 0$$

$$\Leftrightarrow \lambda = 2 \pm \sqrt{4-6} = 2 \pm \sqrt{-2}$$

$$\Leftrightarrow \lambda = 2 \pm \sqrt{2}\, i$$

$$|\lambda| = \sqrt{2^2 + \sqrt{2}^2} = \sqrt{4+2} = \sqrt{6} > 1$$

So the fixed point is a source.

(ii)(b) See Maple

3. a) eigenvalues:

$$(4-\lambda)(3-\lambda)-30=0 \iff \lambda^2 - 7\lambda + (12-30) = 0$$

$$\iff \lambda^2 - 7\lambda - 18 = 0$$

$$\iff \lambda = \frac{7 \pm \sqrt{49+72}}{2}$$

$$\iff \lambda = \frac{7 \pm \sqrt{121}}{2} = \frac{7 \pm 11}{2} = \frac{18}{2}, -$$

$$= 9, -2$$

$\because |\lambda_1| > 1$ and $|\lambda_2| > 1$, the origin is a source.

b) eigenvalues:

$$(1-\lambda)\left(\frac{3}{4}-\lambda\right) - \frac{1}{2}\left(\frac{1}{4}\right) = 0 \Leftrightarrow \text{I}$$

$$\text{I} \Leftrightarrow \lambda^2 - \frac{7}{4}\lambda + \frac{3}{4} - \frac{1}{8} = 0$$

$$\Leftrightarrow 8\lambda^2 - 14\lambda + 5 = 0$$

$$\Leftrightarrow \lambda = \frac{7 \pm \sqrt{49-40}}{8} = \frac{7 \pm \sqrt{9}}{8} = \frac{7 \pm 3}{8}$$

$$= \frac{10}{8}, \frac{4}{8} = \frac{5}{4}, \frac{1}{2}$$

$\therefore |\lambda_1| > 1$ + $|\lambda_2| < 1$, so the origin is a saddle.

c) eigenvalues:

$$(-0.4-\lambda)(1.6-\lambda) - 2.4(-0.4) = 0 \Leftrightarrow \text{II}$$

$$\text{II} \Leftrightarrow \lambda^2 - 1.2\lambda - 0.64 + 0.96 = 0$$

$$\Leftrightarrow \lambda^2 - 1.2\lambda + 0.32 = 0$$

$$\Leftrightarrow \lambda = 0.6 \pm \sqrt{0.36 - 0.32} = 0.6 \pm \sqrt{0.04}$$

$$= 0.6 \pm 0.2 = 0.8, 0.4$$

$\therefore |\lambda_1| < 1$ + $|\lambda_2| < 1$, the origin is a sink.

4. eigenvectors:

Let $\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}$ be the eigenvector. Then the eigenvector associated with $\lambda = \frac{5}{4}$ is given by

$$A\vec{u} = \lambda \vec{u} \iff \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \frac{5}{4} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\iff \begin{cases} u + \frac{v}{2} = \frac{5}{4} u \\ \frac{1}{4} u + \frac{3}{4} v = \frac{5}{4} v \end{cases} \iff \begin{cases} 4u + 2v = 5u \\ u + 3v = 5v \end{cases}$$

$$\iff \begin{cases} -u + 2v = 0 \\ u - 2v = 0 \end{cases}$$ These two equations are redundant.

We select $v = 1$, then $u = 2$ and have the eigenvector

$$\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The eigenvector associated with $\lambda = \frac{1}{2}$ is given by

$$A\vec{u} = \lambda \vec{u} \Leftrightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{2} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} u + \frac{1}{2} v = \frac{u}{2} \\ \frac{1}{4} u + \frac{3}{4} v = \frac{1}{2} v \end{cases}$$

$$\Leftrightarrow \begin{cases} 2u + v = u \\ u + 3v = 2v \end{cases}$$

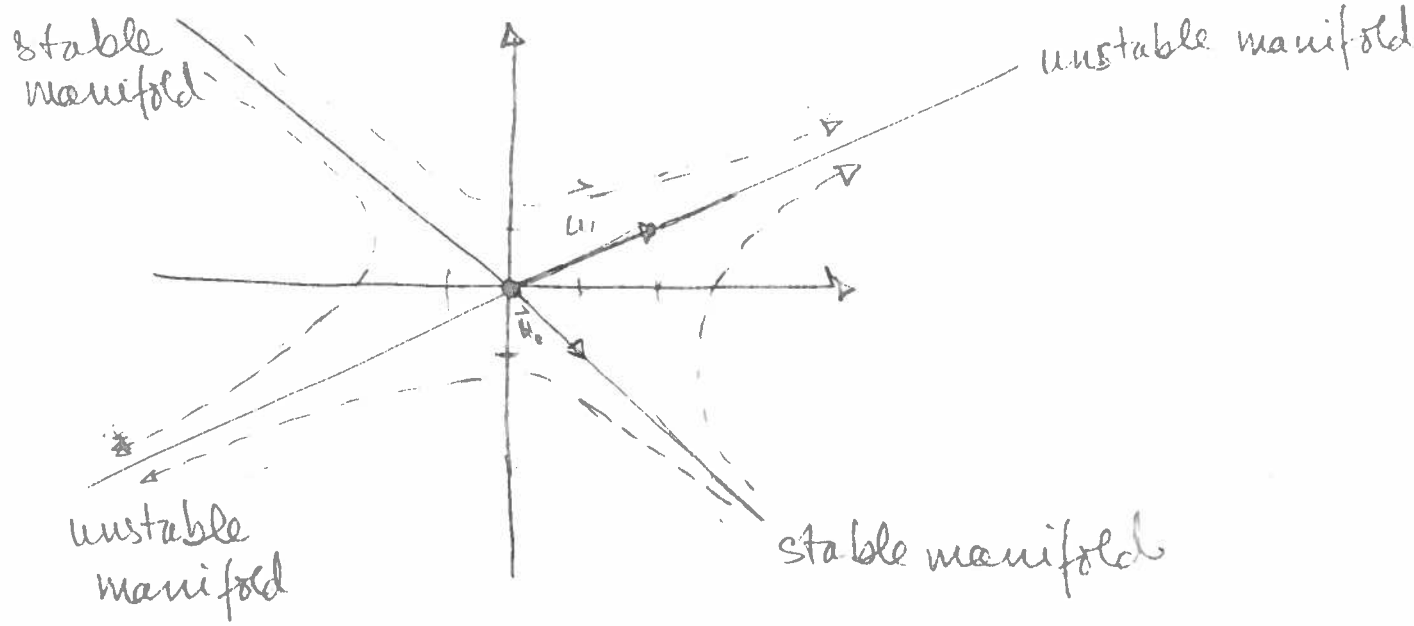
$$\Leftrightarrow \begin{cases} u + v = 0 \\ u + v = 0 \end{cases}$$ These two equations are redundant.

We select $u = 1$, then $v = -1$ and obtain the eigenvector

$$\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So the <u>unstable manifold</u> is the line through $(0,0)$ in direction $\vec{u}_1$, & the <u>stable manifold</u> is the line through $(0,0)$ in direction $\vec{u}_2$.

Sketch:


stable manifold
unstable manifold
$\vec{u}_1$
$\vec{u}_0$
unstable manifold
stable manifold