> $with(plots) : with(LinearAlgebra) :$

Solution to part 1(b).

> $f := (x, y) \rightarrow \left\langle 2\cdot\sin\left(\frac{\text{Pi}\cdot x}{3}\right), \frac{y}{2}\right\rangle;$

$$f := (x, y) \rightarrow \left\langle 2\sin\left(\frac{1}{3}\pi x\right), \frac{1}{2} y\right\rangle \tag{1}$$

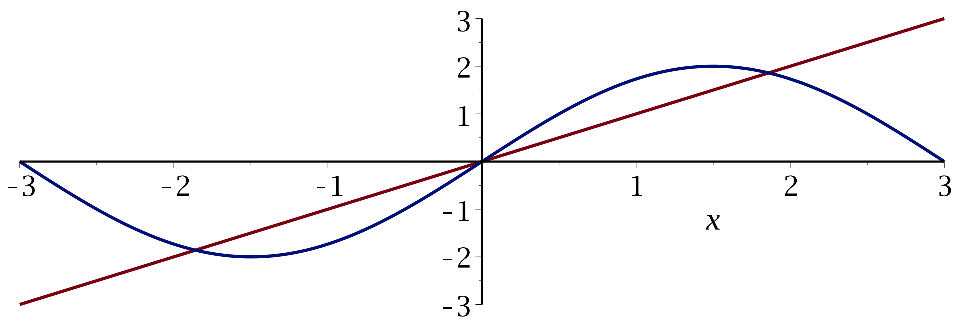
By analytically solving for the steady states (see the handwritten solutions for assignment #2), we see that the steady state value of y is 0. To find the steady state values of x, we need to determine when the function

> $fp := x \rightarrow 2\cdot\sin\left(\frac{\text{Pi}\cdot x}{3}\right);$

$$fp := x \rightarrow 2\sin\left(\frac{1}{3}\pi x\right) \tag{2}$$

is equal to x. Plot fp(x) and x in order to determine where the intersections are.

> $plot([x, fp(x)], x = -3..3)$



We see that one of the steady state values of x is 0. The other two steady state values are near -2 and 2. We use this information to guide our choice of starting points for the root-solving routine.

> $fsolve(fp(x) = x, x = -2..-1); fsolve(fp(x) = x, x = 1..2);$

$$-1.859747547$$

$$1.859747547 \tag{3}$$

So the fixed points are (0,0), (-1.86,0), and (1.86,0). Now we look at the image of the unit circle under two iterations of the map f at these three steady states.

We begin by looking at the image of a small circle centred at the origin.

> $x0vec := \langle 0.1\cdot\cos(t), 0.1\cdot\sin(t)\rangle;$

$$x0vec := \begin{bmatrix} 0.1\cos(t) \\ 0.1\sin(t) \end{bmatrix} \tag{4}$$

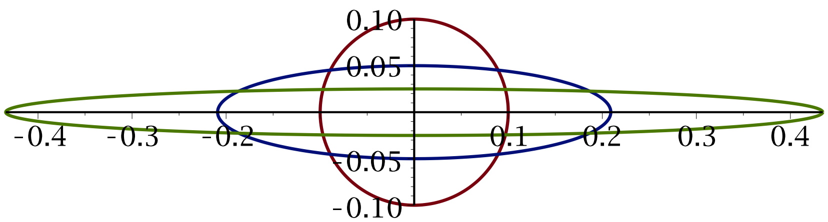
> $x1vec := f(x0vec[1], x0vec[2]);$

$$x1vec := \begin{bmatrix} 2\sin(0.03333333333\,\pi\cos(t)) \\ 0.05000000000\sin(t) \end{bmatrix} \tag{5}$$

> $x2vec := f(x1vec[1], x1vec[2]);$

$$x2vec := \begin{bmatrix} 2\sin\left(\frac{2}{3}\pi\sin(0.03333333333\,\pi\cos(t))\right) \\ 0.02500000000\sin(t) \end{bmatrix} \quad (6)$$

> $plot([[x0vec[1], x0vec[2], t = 0..2\cdot Pi], [x1vec[1], x1vec[2], t = 0..2\cdot Pi], [x2vec[1], x2vec[2], t = 0..2\cdot Pi]], scaling = constrained);$



The colour ordering is: red = initial circle, blue = first iterate, green = second iterate. This figure shows that the initial circle is transformed into ellipses that elongate along the x axis and shrink along the y axis. So the steady state at (0,0) is a saddle.

Now we look at the steady state at (-1.86,0).

> $x0vec := \langle 0.1\cdot\cos(t) - 1.86, 0.1\cdot\sin(t)\rangle;$

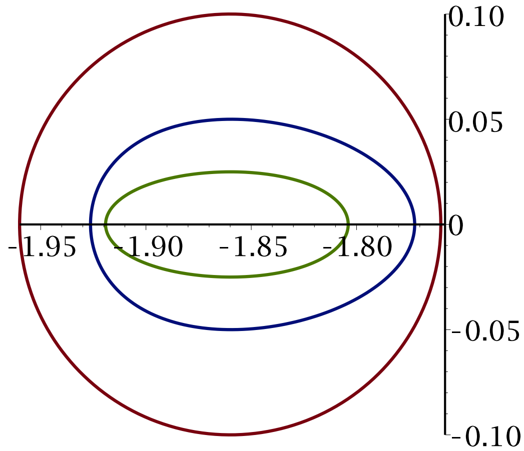
$$x0vec := \begin{bmatrix} 0.1\cos(t) - 1.86 \\ 0.1\sin(t) \end{bmatrix} \quad (7)$$

> $x1vec := f(x0vec[1], x0vec[2]); x2vec := f(x1vec[1], x1vec[2]);$

$$x1vec := \begin{bmatrix} 2\sin\left(\frac{1}{3}\pi\,(0.1\cos(t) - 1.86)\right) \\ 0.05000000000\sin(t) \end{bmatrix}$$

$$x2vec := \begin{bmatrix} 2\sin\left(\frac{2}{3}\pi\sin\left(\frac{1}{3}\pi\,(0.1\cos(t) - 1.86)\right)\right) \\ 0.02500000000\sin(t) \end{bmatrix} \quad (8)$$

> $plot([[x0vec[1], x0vec[2], t = 0..2\cdot Pi], [x1vec[1], x1vec[2], t = 0..2\cdot Pi], [x2vec[1], x2vec[2], t = 0..2\cdot Pi]], scaling = constrained);$



Here we see that the initial circle (red) becomes successively smaller in all directions (albeit at different rates) under forward iterations of the map. So the steady state at (-1.86,0) is a sink.

Now we consider the last steady state at (1.86,0).

> $x0vec := \langle 0.1\cdot\cos(t)+1.86, 0.1\cdot\sin(t)\rangle$

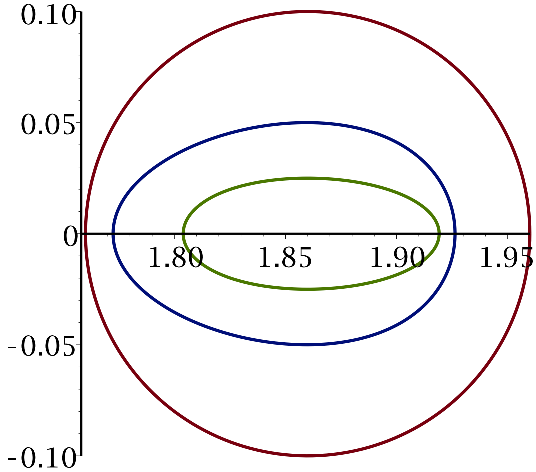
$$x0vec := \begin{bmatrix} 0.1\cos(t)+1.86 \\ 0.1\sin(t) \end{bmatrix} \quad \textbf{(9)}$$

> $x1vec := f(x0vec[1], x0vec[2]);\ x2vec := f(x1vec[1], x1vec[2]);$

$$x1vec := \begin{bmatrix} 2\sin\left(\frac{1}{3}\pi\left(0.1\cos(t)+1.86\right)\right) \\ 0.05000000000\sin(t) \end{bmatrix}$$

$$x2vec := \begin{bmatrix} 2\sin\left(\frac{2}{3}\pi\sin\left(\frac{1}{3}\pi\left(0.1\cos(t)+1.86\right)\right)\right) \\ 0.02500000000\sin(t) \end{bmatrix} \quad \textbf{(10)}$$

> $plot([[x0vec[1], x0vec[2], t=0..2\cdot\text{Pi}], [x1vec[1], x1vec[2], t=0..2\cdot\text{Pi}], [x2vec[1], x2vec[2], t=0..2\cdot\text{Pi}]], scaling=constrained);$



This result is the same as the one obtained for the mirror steady state at (-1.86,0).
We thus conclude that the steady state at (1.86,0) is a sink.

Solution to part 2(ii).

Problem 2(a)

> $A := \left\langle \langle 2, 2 \rangle \middle| \left\langle \frac{1}{2}, -\frac{1}{2} \right\rangle \right\rangle;$

$$A := \begin{bmatrix} 2 & \frac{1}{2} \\ 2 & -\frac{1}{2} \end{bmatrix} \quad \textbf{(11)}$$

> $x0vec := \langle \cos(t), \sin(t) \rangle;$

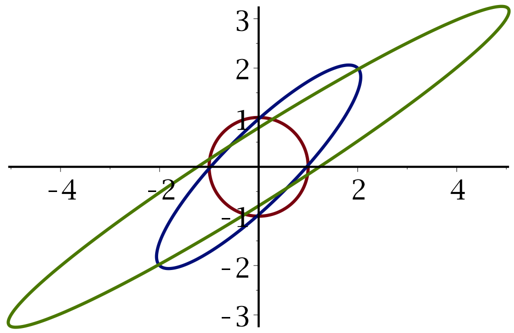
$$x0vec := \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \quad \textbf{(12)}$$

> $x1vec := A\, x0vec;\ x2vec := A\, x1vec;$

$$x1vec := \begin{bmatrix} 2\cos(t) + \frac{1}{2}\sin(t) \\ 2\cos(t) - \frac{1}{2}\sin(t) \end{bmatrix}$$

$$x2vec := \begin{bmatrix} 5\cos(t) + \frac{3}{4}\sin(t) \\ 3\cos(t) + \frac{5}{4}\sin(t) \end{bmatrix} \quad \textbf{(13)}$$

> $plot([[x0vec[1], x0vec[2], t = 0..2 \cdot \mathrm{Pi}], [x1vec[1], x1vec[2], t = 0..2 \cdot \mathrm{Pi}], [x2vec[1], x2vec[2], t = 0..2 \cdot \mathrm{Pi}]], scaling = constrained)$



We see that the unit circle is being stretched in one direction (approximately along the line through (0,0) with slope 1), and shrunk in the other. So the steady state at (0,0) is a saddle.

We note that the eigenvalues of the matrix A are

> *Eigenvalues*(*A*);

$$\begin{bmatrix} \frac{3}{4} + \frac{1}{4}\sqrt{41} \\ \frac{3}{4} - \frac{1}{4}\sqrt{41} \end{bmatrix} \quad \textbf{(14)}$$

These eigenvalues are both real, one larger than one and the other less than one. This confirms our classification as correct.

Problem 2(b)

> $A := \langle\langle 2,-2\rangle|\langle 1, 2\rangle\rangle;$

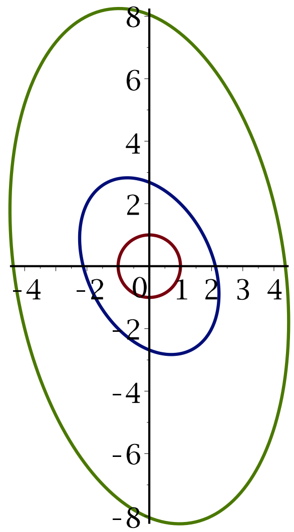
$$A := \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix} \quad \textbf{(15)}$$

> $x1vec := A\, x0vec;\ x2vec := A\, x1vec;$

$$x1vec := \begin{bmatrix} 2\cos(t) + \sin(t) \\ -2\cos(t) + 2\sin(t) \end{bmatrix}$$

$$x2vec := \begin{bmatrix} 2\cos(t) + 4\sin(t) \\ -8\cos(t) + 2\sin(t) \end{bmatrix} \quad \textbf{(16)}$$

> *plot*([[*x0vec*[1], *x0vec*[2], *t* = 0 ..2·Pi], [*x1vec*[1], *x1vec*[2], *t* = 0 ..2·Pi], [*x2vec*[1], *x2vec*[2], *t* = 0 ..2·Pi]], *scaling* = *constrained*)



In this case we see that the initial circle is growing in all directions, and so the origin is a source. The eigenvalues of this matrix are

> *Eigenvalues*(*A*);

$$\begin{bmatrix} 2+\mathrm{I}\sqrt{2} \\ 2-\mathrm{I}\sqrt{2} \end{bmatrix} \quad \textbf{(17)}$$

with magnitude

> |*Eigenvalues*(*A*)|;

$$\begin{bmatrix} \sqrt{6} \\ \sqrt{6} \end{bmatrix} \quad \textbf{(18)}$$

The eigenvalues thus have magnitude larger than 1, confirming that the origin is a source. The eigenvalues are complex however, and so the map expands and rotates the ellipse at each iteration.

>