A#3. Sol'ns

1. $f(x,y,z) = \left(x^2y,\ y^4,\ \frac{xz}{2} + y\right)$

Fixed pts.

$$\begin{cases} x^* = x^{*2}y^* & \dots\dots (1) \\ y^* = y^{*4} & \dots\dots (2) \\ z^* = \frac{x^*z^*}{2} + y^* & \dots\dots (3) \end{cases}$$

By (2) we have

$$y^* = 0 \text{ or } y^{*3} = 1 \Leftrightarrow y^* = 1.$$

If $y^* = 0$ then by (1) we have $x^* = 0$ and by (3) we have $z^* = 0$.

If $y^* = 1$, then by (1) we have

$$x^* = x^{*2} \Leftrightarrow x^* = 0 \text{ or } x^* = 1.$$

If $x^* = 0$ then by (3) we have

$$z^* = y^* = 1.$$

If $x^* = 1$ then by (3) we have

$$z^* = \frac{z^*}{2} + y^* \Leftrightarrow z^* = \frac{z^*}{2} + 1 \Leftrightarrow z^* = 2.$$

So the fixed points are:

$$(0,0,0);\ (0,1,1);\ (1,1,2)$$

<u>Stability</u>

$$Df(x,y,z) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}$$

where $f_1 = x^2 y$, $f_2 = y^4$, $f_3 = \frac{xz}{2} + y$. This gives us

$$Df(\vec{x}) = \begin{bmatrix} 2xy & x^2 & 0 \\ 0 & 4y^3 & 0 \\ \frac{z}{2} & 1 & \frac{x}{2} \end{bmatrix}.$$

We evaluate this matrix at each steady state & find the eigenvalues.

Stability of fixed point (0,0,0).

$$Df(2,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = 0$

$\therefore |\lambda_i| < 1$ and $(0,0,0)$ is a sink.

Stability of fixed point (0,1,1):

$$Df(0,1,1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ \frac{1}{2} & 1 & 0 \end{bmatrix}$$

eigenvalues: $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 4$

$\therefore |\lambda_1| = |\lambda_2| = 0 < 1$, & $|\lambda_3| > 1$,

so $(0,1,1)$ is a saddle.

Stability of fixed point $(1,1,2)$:

$$Df(1,1,2) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix}$$

eigenvalues: $\lambda_1 = 2,\ \lambda_2 = 4,\ \lambda_3 = \frac{1}{2}$

$\therefore\ |\lambda_1| > 1$ & $|\lambda_2| > 1$, & $|\lambda_3| < 1$

so $(1,1,2)$ is a saddle.

2. $f(x,y) = (ax - bx^2,\ x^2 + 2y)$

a) $a = 4,\ b = 1$

Let us write the given period 2 orbit as

$$\{\vec{p}_1, \vec{p}_2\} = \left\{ \left(\frac{5+\sqrt{5}}{2},\ -\frac{15}{2} - \frac{5\sqrt{5}}{6} \right), \left(\frac{5-\sqrt{5}}{2},\ -\frac{15}{2} + \frac{5\sqrt{5}}{6} \right) \right\}.$$

of the given points do indeed form a period-2 orbit, then we should find that $f(\vec{p}_1) = \vec{p}_2$ + $f(\vec{p}_2) = \vec{p}_1$.

Verification

$$f(\vec{p}_1) = \left(4\left(\frac{5+\sqrt{5}}{2}\right) - \left[\frac{(5+\sqrt{5})}{2}\right]^2, \left[\frac{(5+\sqrt{5})}{2}\right]^2 + \cancel{2}\left(\frac{-15}{\cancel{2}} - \frac{5\sqrt{5}}{\cancel{6}\,3}\right)\right)$$

$$= \left(\frac{40+8\sqrt{5} - (25+10\sqrt{5}+5)}{4}, \frac{25+10\sqrt{5}+5}{4} - 15 - \frac{5\sqrt{5}}{3}\right)$$

$$= \left(\frac{10-2\sqrt{5}}{4}, \frac{15+5\sqrt{5}}{2} - 15 - \frac{5\sqrt{5}}{3}\right)$$

$$= \left(\frac{5-\sqrt{5}}{2}, -\frac{15}{2} + \frac{15\sqrt{5}-10\sqrt{5}}{6}\right)$$

$$= \left(\frac{5-\sqrt{5}}{2}, -\frac{15}{2} + \frac{5\sqrt{5}}{6}\right) = \vec{p}_2$$

$$f(\vec{p}_2) = \left(4\left(\frac{5-\sqrt{5}}{2}\right) - \frac{(5-\sqrt{5})^2}{4}, \frac{(5-\sqrt{5})^2}{4} + 2\left(-\frac{15}{2} + \frac{5\sqrt{5}}{6}\right)\right)$$

$$= \left(\frac{40-8\sqrt{5} - (25-10\sqrt{5}+5)}{4}, \frac{25-10\sqrt{5}+5}{4} - 15 + \frac{5\sqrt{5}}{3}\right)$$

$$= \left(\frac{5+\sqrt{5}}{2}, -\frac{15}{2} - \frac{5\sqrt{5}}{6}\right) = \vec{p}_1$$

$\therefore$ the orbit $\{\vec{p}_1, \vec{p}_2\}$ is a period-2 orbit for f.

<u>Stability</u>

$$Df(x,y) = \begin{bmatrix} 4-2x & 0 \\ 2x & 2 \end{bmatrix}$$

$$Df^2(\vec{p}_1) = Df(\vec{p}_2) \cdot Df(\vec{p}_1)$$

$$= \begin{bmatrix} 4-\cancel{2}\left(\frac{5-\sqrt{5}}{\cancel{2}}\right) & 0 \\ \cancel{2}\left(\frac{5-\sqrt{5}}{\cancel{2}}\right) & 2 \end{bmatrix} \begin{bmatrix} 4-\cancel{2}\left(\frac{5+\sqrt{5}}{\cancel{2}}\right) & 0 \\ \cancel{2}\left(\frac{5+\sqrt{5}}{\cancel{2}}\right) & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1+\sqrt{5} & 0 \\ 5-\sqrt{5} & 2 \end{bmatrix} \begin{bmatrix} -1-\sqrt{5} & 0 \\ 5+\sqrt{5} & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-5 & 0 \\ 10-2\sqrt{5} & 4 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 10-2\sqrt{5} & 4 \end{bmatrix}$$

$$\lambda_1 = 4, \quad \lambda_2 = -4$$

$\therefore$ the period-2 orbit is a <u>source</u>.

b) For $a = \frac{1}{3}$ & $b = 0$, the map is

$$f(x,y) = \left(\tfrac{1}{3}x,\ x^2 + 2y\right).$$

(0,0) is a fixed point, & we wish to determine its stability:

$$Df(x,y) = \begin{bmatrix} \frac{1}{3} & 0 \\ 2x & 2 \end{bmatrix}$$

$$\therefore Df(0,0) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}$$

The eigenvalues are

$$\lambda_1 = \tfrac{1}{3}, \text{ so } |\lambda_1| < 1$$

$$\lambda_2 = 2, \text{ so } |\lambda_2| > 1$$

$\therefore$ the origin is a <u>saddle</u>.

3. $f(x,y) = (ax, bx^3 + cy)$

a) $f(0,0) = (0,0)$, $\therefore (0,0)$ is a fixed point.

$$Df(x,y) = \begin{bmatrix} a & 0 \\ 3bx^2 & c \end{bmatrix} \quad \therefore Df(0,0) = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$$

The eigenvalues are $\lambda_1 = a$ & $\lambda_2 = c$. $\therefore$ the origin is a saddle if

- $|a| < 1$ & $|c| > 1$

OR

- $|a| > 1$ & $|c| < 1$

b)
$$\begin{cases} x_{n+1} = a x_n \\ y_{n+1} = b x_n^3 + c y_n \end{cases} \Leftrightarrow \begin{cases} x_n = \frac{1}{a} x_{n+1} \\ y_n = \frac{1}{c}\left(y_{n+1} - b x_n^3\right) \end{cases}$$

$$\Leftrightarrow \begin{cases} x_n = \frac{1}{a} x_{n+1} \\ y_n = \frac{1}{c}\left(y_{n+1} - \frac{b}{a^3} x_{n+1}^3\right) \end{cases}$$

$$\therefore f^{-1}(x,y) = \left(\frac{1}{a}x, \ \frac{1}{c}\left(y - \frac{b}{a^3}x^3\right)\right)$$

c) $f(t, qt^3) = (at, bt^3 + cqt^3)$

$$= (at, (b+qc)t^3)$$

$\therefore$ the curve $y = qx^3$ is forward invariant if

$$(b+qc) = qa^3 \iff q(a^3 - c) = b$$
$$\iff q = \frac{b}{a^3 - c}.$$

Similarly,

$$f^{-1}(t, qt^3) = \left(\frac{1}{a}t, \frac{1}{c}\left(qt^3 - \frac{b}{a^3}t^3\right)\right)$$
$$= \left(\frac{t}{a}, \left(q - \frac{b}{a^3}\right)\frac{1}{c}t^3\right)$$

$\therefore$ the curve $y = qx^3$ is backward invariant if

$$\left(q - \frac{b}{a^3}\right)\frac{1}{c} = \frac{q}{a^3} \iff q\left(\frac{1}{c} - \frac{1}{a^3}\right) = \frac{b}{ca^3} \iff$$

$$\iff q(a^3 - c) = b \iff q = \frac{b}{a^3 - c}.$$

Hence, S is invariant.

d) The eigenvectors of $Df(0,0)$ are $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. We see that the first of these is tangent to $\mathcal{S}$ at the origin, and [illegible]

$$f^{n}(t, gt^3) \to 0 \text{ as } n \to \infty \text{ if } |a|<1,$$

$$f^{-n}(t, gt^3) \to 0 \text{ as } n \to \infty \text{ if } |a|>1.$$

So $\mathcal{S}$ is one of the manifolds of f. The other eigenvector points along the y axis, so we test points on the y axis to see if that set is invariant:

$$f(0,y) = (0, cy) \to \text{still on the } y \text{ axis,}$$

$\therefore$ <u>forward</u> invariant

$$f^{-1}(0,y) = \left(0, \frac{y}{c}\right) \to \text{still on the } y \text{ axis,}$$

$\therefore$ <u>backward</u> invariant

Hence, the y axis is invariant. Also,

- $f^{n}(0,y) \to 0$ as $n \to \infty$ if $|c|<1$
- $f^{-n}(0,y) \to 0$ as $n \to \infty$ if $|c|>1$

$\therefore$ we have two cases.

1) $|a| < 1$, $|c| > 1$

$$\mathcal{S} = \{(x, qx^3) : x \in \mathbb{R}\}$$

$$\mathcal{U} = \{(0, y) : y \in \mathbb{R}\}$$

2) $|a| > 1$, $|c| < 1$

$$\mathcal{S} = \{(0, y) : y \in \mathbb{R}\}$$

$$\mathcal{U} = \{(x, qx^3) : x \in \mathbb{R}\}$$