Assignment #4

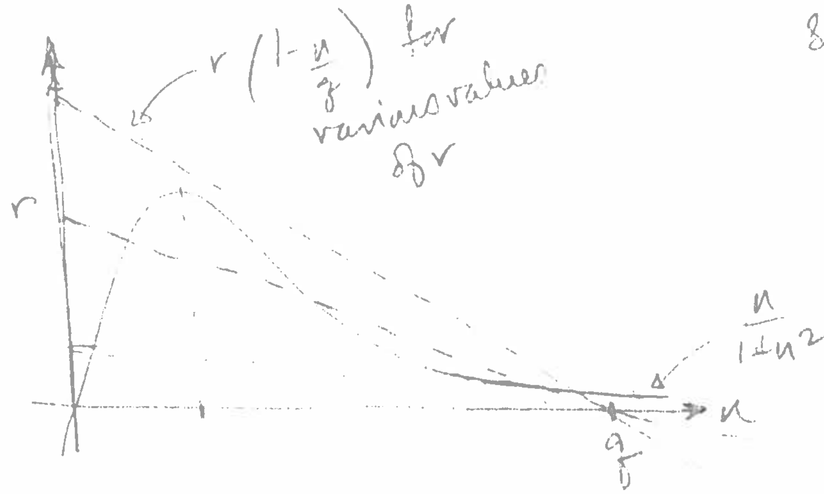
Solution

1. $$\frac{du}{dt} = ru\left(1-\frac{u}{q}\right) - \frac{u^2}{1+u^2}$$

a) $$\frac{du}{dt} = 0 \iff ru\left(1-\frac{u}{q}\right) - \frac{u^2}{1+u^2} = 0$$

$$\iff u\left(r\left(1-\frac{u}{q}\right) - \frac{u}{1+u^2}\right) = 0$$

$$\iff u = 0 \text{ or } \underbrace{r\left(1-\frac{u}{q}\right) = \frac{u}{1+u^2}}_{\text{solve graphically}} \quad \ldots (1)$$



We see that there are three possibilities

i) 1 intersection at u^* near 0

ii) 3 "

iii) 1 " at u^* near q

Stability

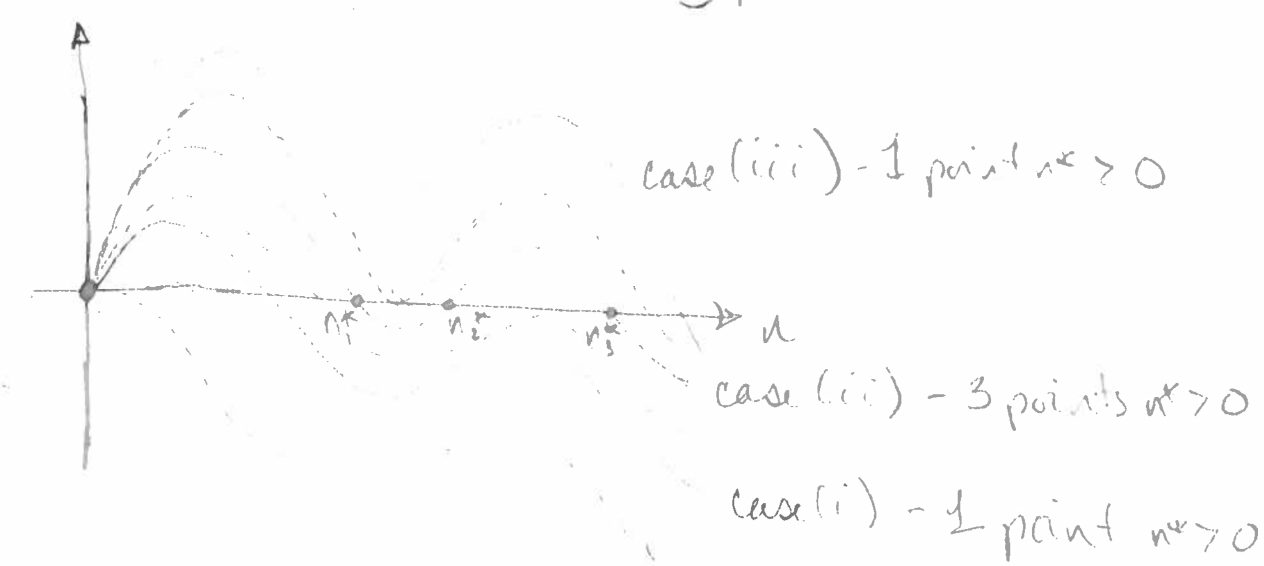
$$f(n) = rn\left(1 - \frac{n}{q}\right) - \frac{n^2}{1+n^2}$$

$$f'(n) = r - \frac{2rn}{q} - \frac{(1+n^2)2n - n^2 \cdot 2n}{(1+n^2)^2}$$

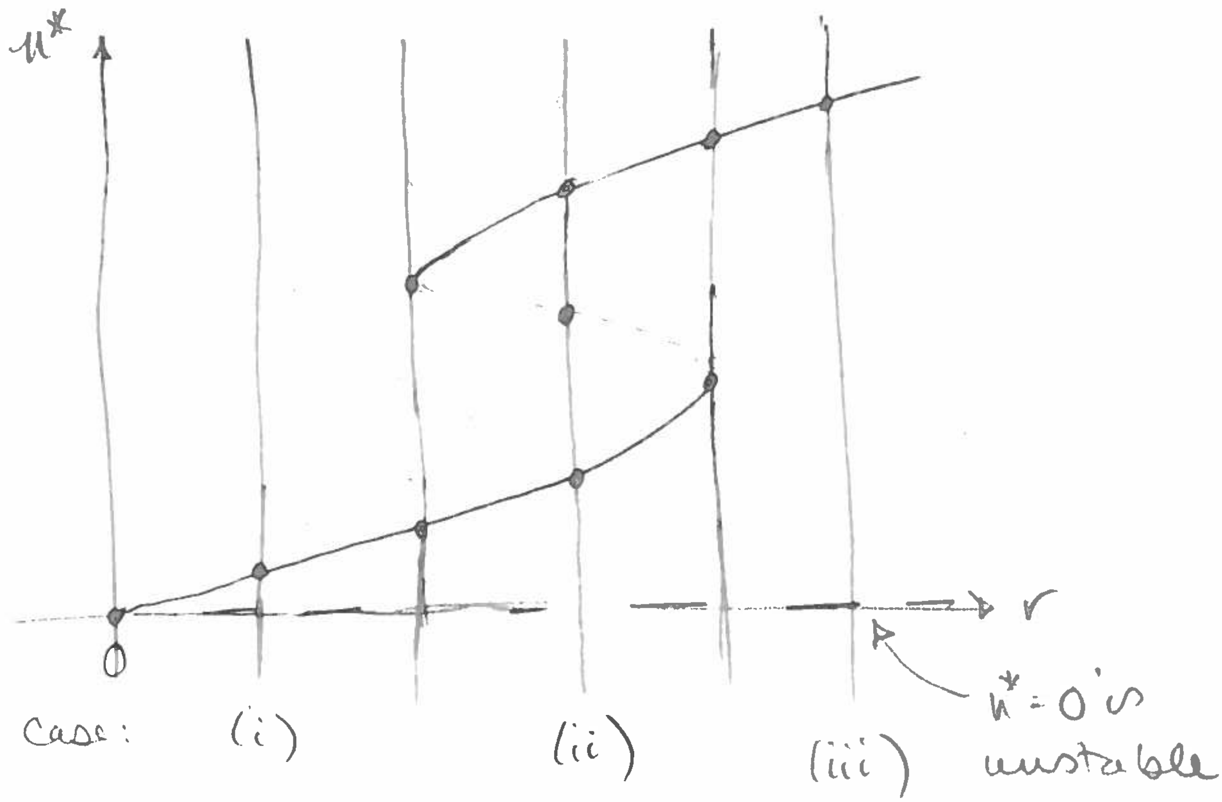
$$= r - \frac{2rn}{q} - \frac{2n}{(1+n^2)^2} \qquad \cdots\cdots \qquad (2)$$

$\therefore f'(0) = r > 0$, and so the $n^* = 0$ steady state is unstable (source).

To find the stability of the other steady states, we again use graphical techniques. $\because f(n)$ & $f'(n)$ are continuous $\forall n \geq 0$, we know that $f(n)$ must follow the following pattern:



Phase Lines



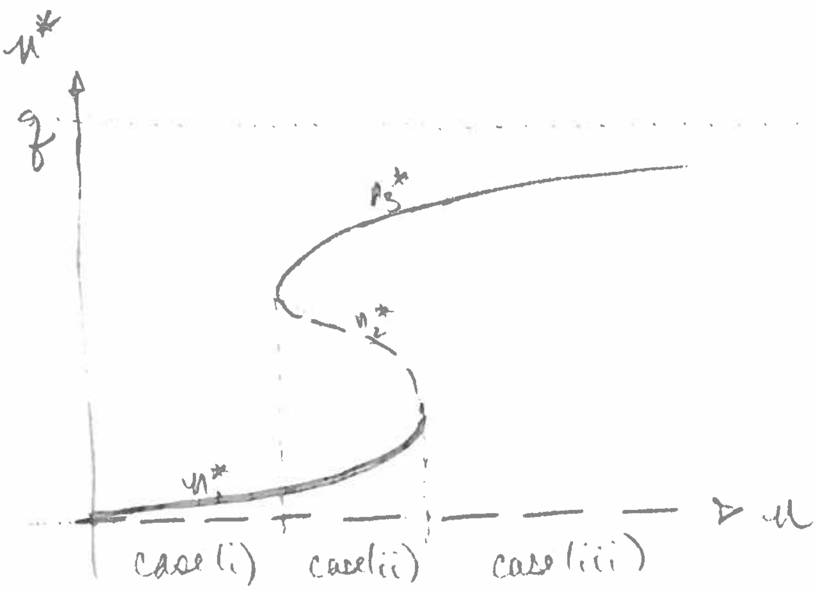
We see that, given $f'(0) > 0$, we must have

case (i) $f'(n^*) < 0$ (for $n^* > 0$), + thus, steady state is stable

case (ii) $f'(n_1^*) < 0$, $f'(n_2^*) > 0$, $f'(n_3^*) < 0$

$\downarrow$ stable (sink) | $\downarrow$ unstable (source) | $\downarrow$ stable (sink)

(c)



2. $2y'' - 7y' + 3y = 0$

Characteristic equation:

$$2r^2 - 7r + 3 = 0 \iff r = \frac{7 \pm \sqrt{49 - 4\cdot 6}}{4} = \frac{7 \pm \sqrt{25}}{4}$$

$$= \frac{7 \pm 5}{4} = \frac{12}{4} \text{ \& } \frac{2}{4} = 3 \text{ \& } \frac{1}{2}$$

The general solution is

$$y(t) = c_1 e^{3t} + c_2 e^{\frac{1}{2}t}$$

3. $y'' + 2y' + y = 0 \quad , \; y(0) = 5, \; y'(0) = -3$

characteristic equation:

$r^2 + 2r + 1 = 0 \iff (r+1)^2 = 0 \iff r = -1$ (double root)

The general solution is

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

Applying the initial condition:

$$\begin{cases} y(0) = 5 \\ y'(0) = -3 \end{cases} \iff \begin{cases} c_1 = 5 \\ -5 + c_2 = -3 \end{cases} \iff \begin{cases} c_1 = 5 \\ c_2 = 2 \end{cases}$$

$\therefore$ the solution is

$$y(t) = 5e^{-t} + 2te^{-t}$$

4. $y'' - 4y' + 5y = 0$

characteristic equation

$$r^2 - 4r + 5 = 0 \Rightarrow r = 2 \pm \sqrt{4-5} = 2 \pm i$$

$$\therefore\ y(t) = e^{2t}\left(c_1 \cos(t) + c_2 \sin(t)\right)$$

5. $x''' - 6x' + (8-x)x = 0$

Let $y_1 = x$, $y_2 = x'$, & $y_3 = x''$. Then the ODE can be written

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = 6y_2 - (8-y_1)y_1. \end{cases}$$