Assignment #5

Solutions

1. a) $\begin{cases} u_1' = u_2 \\ u_2' = -3u_2 + 4u_1 \end{cases} \iff \begin{cases} u_1'' = u_2' \\ u_2' = -3u_1' + 4u_1 \end{cases}$

$\therefore\ u_1'' = -3u_1' + 4u_1 \iff u_1'' + 3u_1' - 4u_1 = 0$

As required.

b) $A = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}$

Eigenvalues:

$|A - \lambda I| = 0 \iff \begin{vmatrix} -\lambda & 1 \\ 4 & -3-\lambda \end{vmatrix} = 0 \iff \lambda^2 + 3\lambda - 4 = 0$

$\iff \lambda = \dfrac{-3 \pm \sqrt{9+16}}{2} = \dfrac{-3 \pm \sqrt{25}}{2}$

$= \dfrac{-3 \pm 5}{2} = \dfrac{2}{2} \text{ or } \dfrac{-8}{2} = 1 \text{ or } -4$

Eigenvectors:

$\lambda_1 = 1$

$(A - \lambda I)\vec{u}_1 = \vec{0} \iff \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} -u_{11} + u_{12} = 0 \\ 4u_{11} - 4u_{12} = 0 \end{cases}$

The 2nd equation is redundant, so we are free to choose one of u_{11} or u_{12}. Let $u_{12} = 1$, then $u_{11} = 1$ and we have $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\underline{\lambda_2 = -4}$

$$(A - \lambda_2 I)\vec{u}_2 = \vec{0} \Leftrightarrow \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} 4u_{21} + u_{22} = 0 \\ 4u_{21} + u_{22} = 0 \end{cases}$$

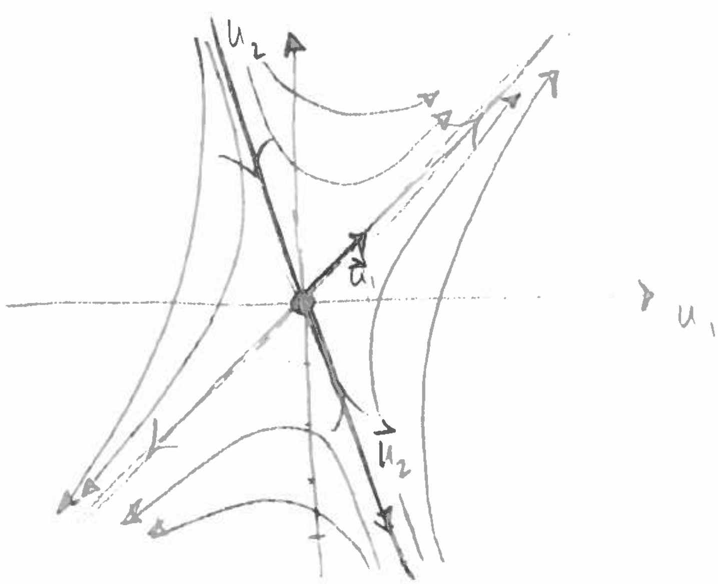
The second equation is redundant, so we are free to choose u_{21} or u_{22}. Let $u_{21} = 1$, then $u_{22} = -4$ and we have

$$\vec{u}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

The solutions are thus

$$\vec{u} = c_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-4t}\begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

c)



$\because \lambda_1 > 0$, $\vec{u}_1$ is the unstable direction.

$\because \lambda_2 < 0$, $\vec{u}_2$ is the stable direction.

The origin is a <u>saddle</u>.

2. $\begin{cases} \dfrac{dx}{dt} = 2x - y \\ \dfrac{dy}{dt} = x^2 + 4y \end{cases}$

Steady states occur at points where

$\begin{cases} x' = 0 \\ y' = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - y = 0 \\ x^2 + 4y = 0 \end{cases} \Leftrightarrow \begin{cases} 2x = y \\ x^2 + 4(2x) = 0 \end{cases} \Leftrightarrow \begin{cases} y = 2x \\ x(x+8) = 0 \end{cases}$

$\Leftrightarrow \begin{cases} y = 0 \text{ or } y = -16 \\ x = 0 \text{ or } x = -8 \end{cases}$

$\therefore$ the steady states are $(0,0)$ and $(-8,-16)$.

Stability:

$J = \begin{bmatrix} 2 & -1 \\ 2x & 4 \end{bmatrix}$

$\underline{(0,0)}$

$J\Big|_{(0,0)} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$, $\lambda_1 = 2$ & $\lambda_2 = 4$

Since both eigenvalues are positive, the steady state is <u>unstable</u>.

$(-8, -16)$

$$J\Big|_{(-8,-16)} = \begin{bmatrix} 2 & -1 \\ -16 & 4 \end{bmatrix}$$

eigenvalues:

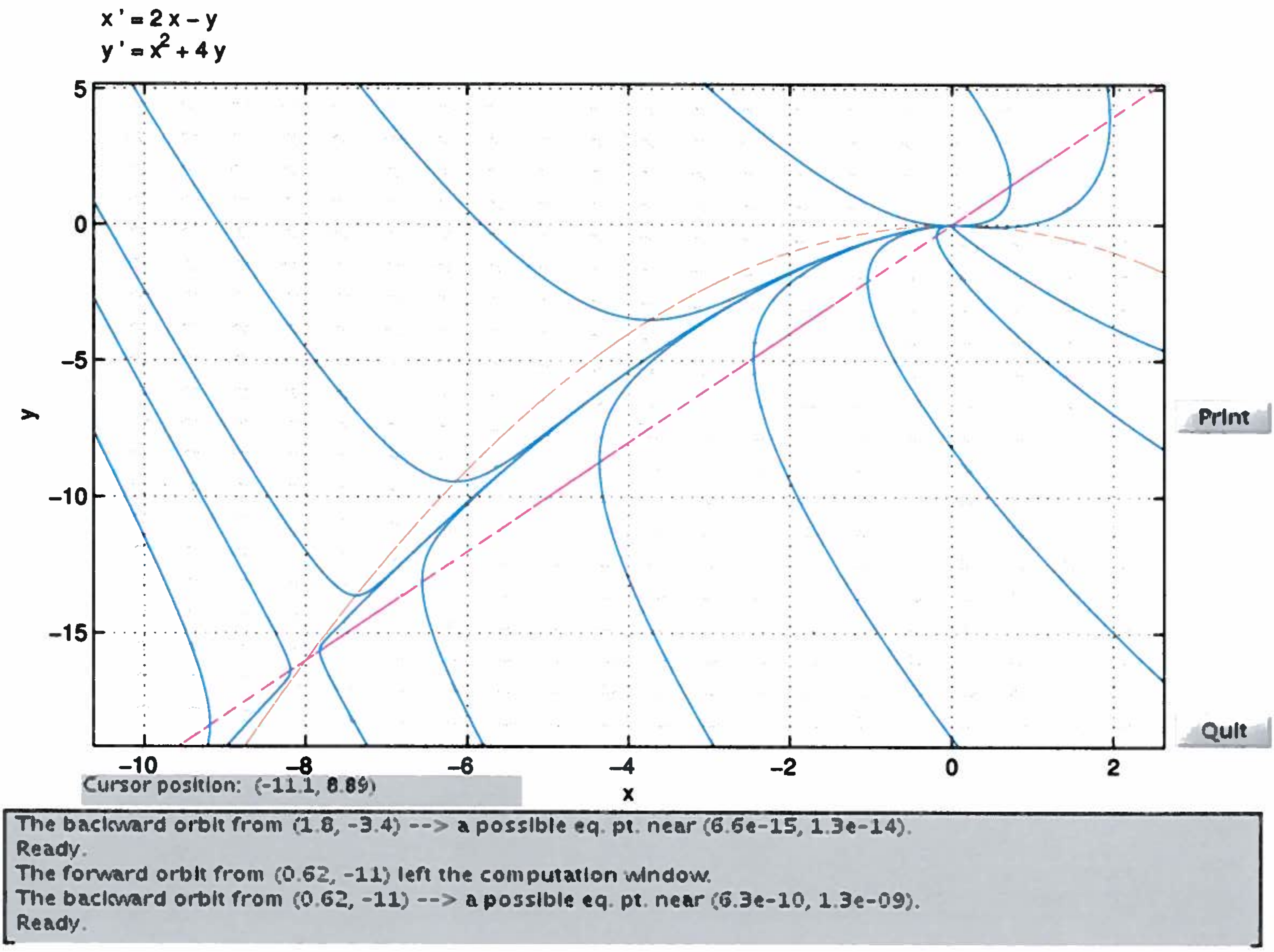
$$(2-\lambda)(4-\lambda) - 16 = 0 \Leftrightarrow \lambda^2 - 6\lambda + 8 - 16 = 0$$
$$\Leftrightarrow \lambda^2 - 6\lambda - 8 = 0$$
$$\Leftrightarrow \lambda = +3 \pm \sqrt{9+8}$$
$$\Leftrightarrow \lambda = +3 \pm \sqrt{17}$$

$\because \sqrt{17} > 4$, we know that $\lambda_+ > 0$, $\lambda_- < 0$

Because one of the eigenvalues is positive, the steady state (8, 16) is unstable.

(Because the other eigenvalue is less than zero, the steady state (-8, -16) is a saddle.)

Phase plane diagrams on next page.


x ' = 2 x − y
y ' = x2 + 4 y
y
x
Print
Quit
Cursor position: (-11.1, 8.89)
The backward orbit from (1.8, -3.4) --> a possible eq. pt. near (6.6e-15, 1.3e-14).
Ready.
The forward orbit from (0.62, -11) left the computation window.
The backward orbit from (0.62, -11) --> a possible eq. pt. near (6.3e-10, 1.3e-09).
Ready.

3. T 7.7

$\dot{\vec{v}} = A\vec{v}$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$f(\vec{v}) = \begin{bmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{bmatrix}$ $f(\vec{w}) = \begin{bmatrix} aw_1 + bw_2 \\ cw_1 + dw_2 \end{bmatrix}$

We seek L such that

$|f(\vec{v}) - f(\vec{w})| \leq L|\vec{v} - \vec{w}| \Leftrightarrow |f(\vec{v}) - f(\vec{w})|^2 \leq L^2 |\vec{v} - \vec{w}|^2$.

$lhs = |f(\vec{v}) - f(\vec{w})|^2$

$= (av_1 + bv_2 - aw_1 - bw_2)^2 + (cv_1 + dv_2 - cw_1 - dw_2)^2$

$= (a(v_1 - w_1) + b(v_2 - w_2))^2 + (c(v_1 - w_1) + d(v_2 - w_2))^2$

$= a^2(v_1 - w_1)^2 + b^2(v_2 - w_2)^2 + 2ab(v_1 - w_1)(v_2 - w_2)$
$+ c^2(v_1 - w_1)^2 + d^2(v_2 - w_2)^2 + 2cd(v_1 - w_1)(v_2 - w_2)$

$= (a^2 + c^2)(v_1 - w_1)^2 + (b^2 + d^2)(v_2 - w_2)^2$
$+ 2(ab + cd)(v_1 - w_1)(v_2 - w_2)$

$\leq \max(a^2 + c^2, b^2 + d^2)|\vec{v} - \vec{w}|^2 + 2|ab + cd|\,|\vec{v} - \vec{w}|^2$

$$\therefore \text{lhs} \leq \left[\max(a^2+c^2, b^2+d^2) + 2|ab+cd|\right] |\vec{v}-\vec{w}|^2$$

Thus, we can choose

$$L = \sqrt{\max(a^2+c^2,\ b^2+d^2) + 2|ab+cd|}$$

and the function $A\vec{v}$ is Lipschitz.

4. 7.13

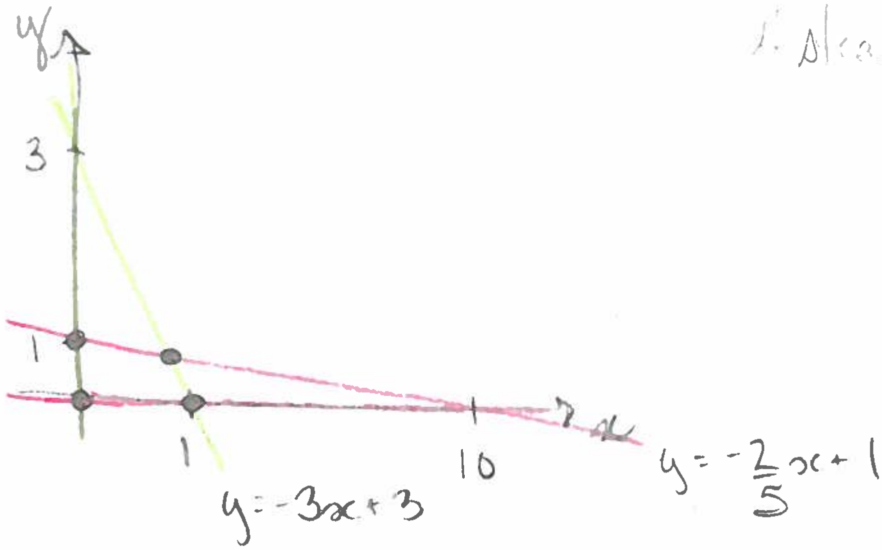
a) $\begin{cases} x' = 3x(1-x) - xy \\ y' = 5y(1-y) - 2xy \end{cases}$

The nullclines are

$$\begin{cases} x' = 0 \\ y' = 0 \end{cases} \Leftrightarrow \begin{cases} 3x(1-x) - xy = 0 \\ 5y(1-y) - 2xy = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \quad \text{or} \quad 3 - 3x - y = 0 \\ y = 0 \quad \text{or} \quad 5 - 5y - 2x = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \quad \text{or} \quad y = -3x + 3 \\ y = 0 \quad \text{or} \quad y = -\frac{2}{5}x + 1 \end{cases}$$



b) Fixed points:

$(0,0)$; $(0,1)$; $(1,0)$

also the intersection of

$$\begin{cases} y = -3x+3 \\ y = -\frac{2}{5}x + 1 \end{cases} \Leftrightarrow \begin{cases} y = -3x+3 \\ -3x+3 = -\frac{2}{5}x+1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = -3x+3 \\ -\frac{13}{5}x = -2 \end{cases} \Leftrightarrow \begin{cases} y = \frac{-30}{13} + \frac{39}{13} = \frac{9}{13} \\ x = \frac{10}{13} \end{cases}$$

∴ the 4th fixed point is

$$\left(\frac{10}{13}, \frac{9}{13}\right)$$

This is the <u>coexistence</u> steady state.

c) Stability

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 3-6x-y & -x \\ -2y & 5-10y-2x \end{bmatrix}$$

<u>at (0,0)</u>

$$J\Big|_{(0,0)} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

$\lambda_1 = 3,\ \lambda_2 = 5$

This steady state is an <u>unstable</u> <u>node</u>.

<u>at (1,0)</u>

$$J\Big|_{(1,0)} = \begin{bmatrix} 3-6 & -1 \\ 0 & 5-2 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 0 & 3 \end{bmatrix}$$

$\lambda_1 = -3,\ \lambda_2 = 3$, $\therefore$ this steady state is a <u>saddle</u> <u>node</u> (unstable).

at $(0,1)$

$$J\Big|_{(0,1)} = \begin{bmatrix} 3-1 & 0 \\ -2 & 5-10 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -2 & 5 \end{bmatrix}$$

$\therefore \lambda_1 = -2$ & $\lambda_2 = 5$, and this steady state is a saddle node (unstable).

at $\left(\frac{10}{13}, \frac{9}{13}\right)$

$$J\Big|_{\left(\frac{10}{13},\frac{9}{13}\right)} = \begin{bmatrix} 3 - \frac{6\cdot 10}{13} - \frac{9}{13} & -\frac{10}{13} \\ -2\cdot\frac{9}{13} & 5 - 10\cdot\frac{9}{13} - 2\cdot\frac{10}{13} \end{bmatrix}$$

$$= \begin{bmatrix} 3 - \frac{69}{13} & -\frac{10}{13} \\ -\frac{18}{13} & 5 - \frac{110}{13} \end{bmatrix}$$

Using maple we obtain

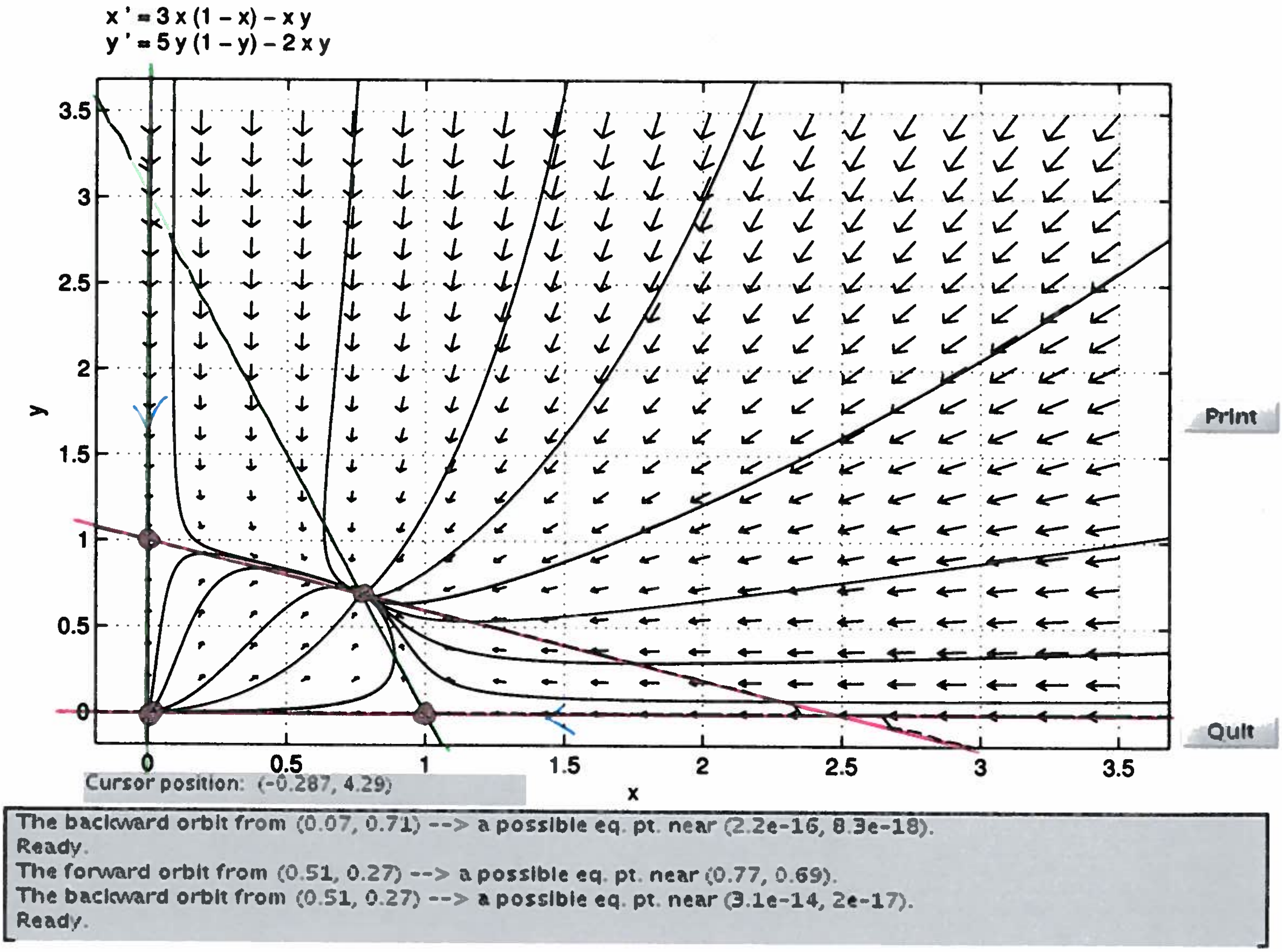
$$\lambda_1 = -\frac{75}{26} + \frac{3\sqrt{105}}{26} = -1.7$$

$$\lambda_2 = -\frac{75}{26} - \frac{3\sqrt{105}}{26} = -4.1$$

$\because$ both eigenvalues are negative, the steady state is <u>stable</u>.

Phase Plane on next page

d) We see that the trajectories all move away from the fixed points (0,0), (1,0), and (0,1), and towards the coexistence state at $\left(\frac{10}{13}, \frac{9}{13}\right)$. Thus, all initial conditions lead to coexistence, & this steady state is globally asymptotically stable (GAS).



e) ∵ the coexistence state is GAS, the outcome is coexistence.