Math 339 - Dynamical Systems
Assignment # 6
due Wed Nov 4th, 12:30pm

Instructions: You are being evaluated on the presentation, as well as the correctness, of your answers. Try to answer questions in a clear, direct, and efficient way. Sloppy or incorrect use of technical terms will lower your mark.

1. Consider the Duffing equation
$$\ddot{x} - x^3 + x = 0. \tag{1}$$
We claimed in class that this equation has an energy function with a double-well potential. This means that solutions should be closed orbits in the phase plane, and that there should be two stable steady states. Verify this claim by following the steps below.
   (a) Convert (1) to Normal Form.
   (b) Find the fixed points.
   (c) Determine the stability of each of the fixed points using the Jacobian.
   (d) Plot the level curves of the corresponding energy function (derived in class), and also plot the phase plane.
   (e) Explain why your results are consistent with the claims we made in class.

2. (Exercise 7.11 p 322) Let $\ddot{x} + b\dot{x} + f(x) = 0$, where $b > 0$ and $f(x) = P'(x)$ and $P(x)$ is shown in figure 7.26 (text p 323). Write the ODE in Normal Form and sketch the phase plane.

3. Consider the system
$$\begin{aligned} \dot{x} &= -z + \left(\frac{x^3}{3} - x\right), && \text{(2a)} \\ \dot{z} &= x. && \text{(2b)} \end{aligned}$$
   (a) Find the steady states of (2).
   (b) Show that $V(x, z) = x^2 + z^2$ is a Lyapunov function for the system (2).
   (c) Find the basin of attraction of the origin.
   (d) What can you conclude from this about the solutions of (2)?
   (e) Use pplane7 to plot the phase plane of (2). Is it consistent with your results above? Explain.

4. Consider the predator-prey system
$$\begin{aligned} \dot{x} &= x(1 - x) - 2xy(1 + x), && \text{(3a)} \\ \dot{y} &= axy - by, && \text{(3b)} \end{aligned}$$
where $a > 0$ and $b > 0$. The term $-2xy$ is due to direct predation effects (being eaten), while the term $-2x^2y$ is due to indirect predation effects (e.g., being frightened all the time and, say, not eating enough). Explore the behaviour of this model as the ratio a/b is varied, as follows:

(a) Find the nullclines and steady states. Sketch these in the phase plane.

(b) Determine the stability of each of the steady states.

(c) If $a = 1$ and b decreases from 1, for what value of b does the coexistence steady state become a focus?

(d) Explore the behaviour of the model using pplane and $a = 1$. Verify that the numerical plots are consistent with your results above.