A#6 Sol'ns

1. ~~Duffing~~ Equ: (Note: I mistyped the equ, so the problem I assigned is not quite the one we did in class.)

a) Normal Form

$$\begin{cases} \dot{x} = y \\ \dot{y} = +x^3 - x \end{cases}$$

b) fixed points

$$\begin{cases} y = 0 \\ -x^3 + x = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x(1-x^2) = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0, \pm 1 \end{cases}$$

$\therefore$ The fixed points are $(0,0)$; $(1,0)$; $(-1,0)$

c) Stability

$$J = \begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{bmatrix}$$

$$|J - \lambda I| = 0 \Leftrightarrow \lambda^2 + 1(-3x^2 + 1) = 0$$

$$\Leftrightarrow \lambda^2 = -1 + 3x^2$$

@ $(0,0)$ $\quad \lambda^2 = -1 \Leftrightarrow \lambda = \pm i \Rightarrow$ centre

@ $(\pm 1, 0)$ $\quad \lambda^2 = -1 + 3 \Leftrightarrow \lambda^2 = +2 \Leftrightarrow \lambda^2 = \pm\sqrt{2}$

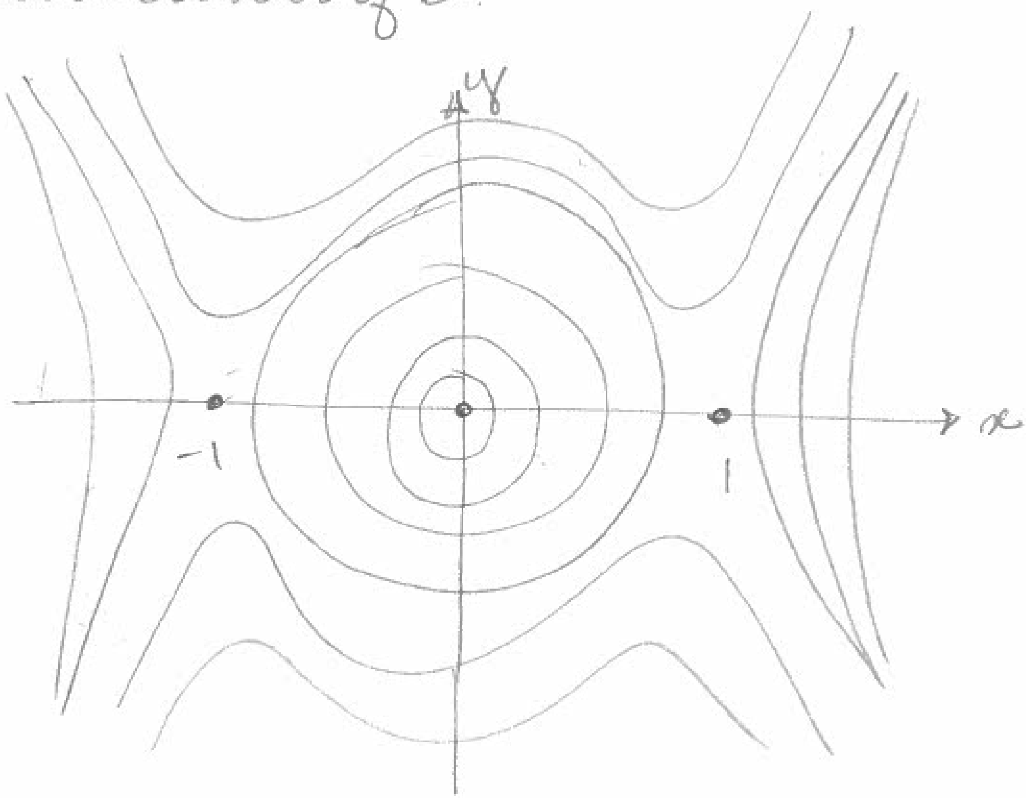
saddles

d) Phase plane & cst energy curves

$$E_1 = \frac{1}{2}(\dot{x})^2 - \frac{x^4}{4} + \frac{x^2}{2}$$

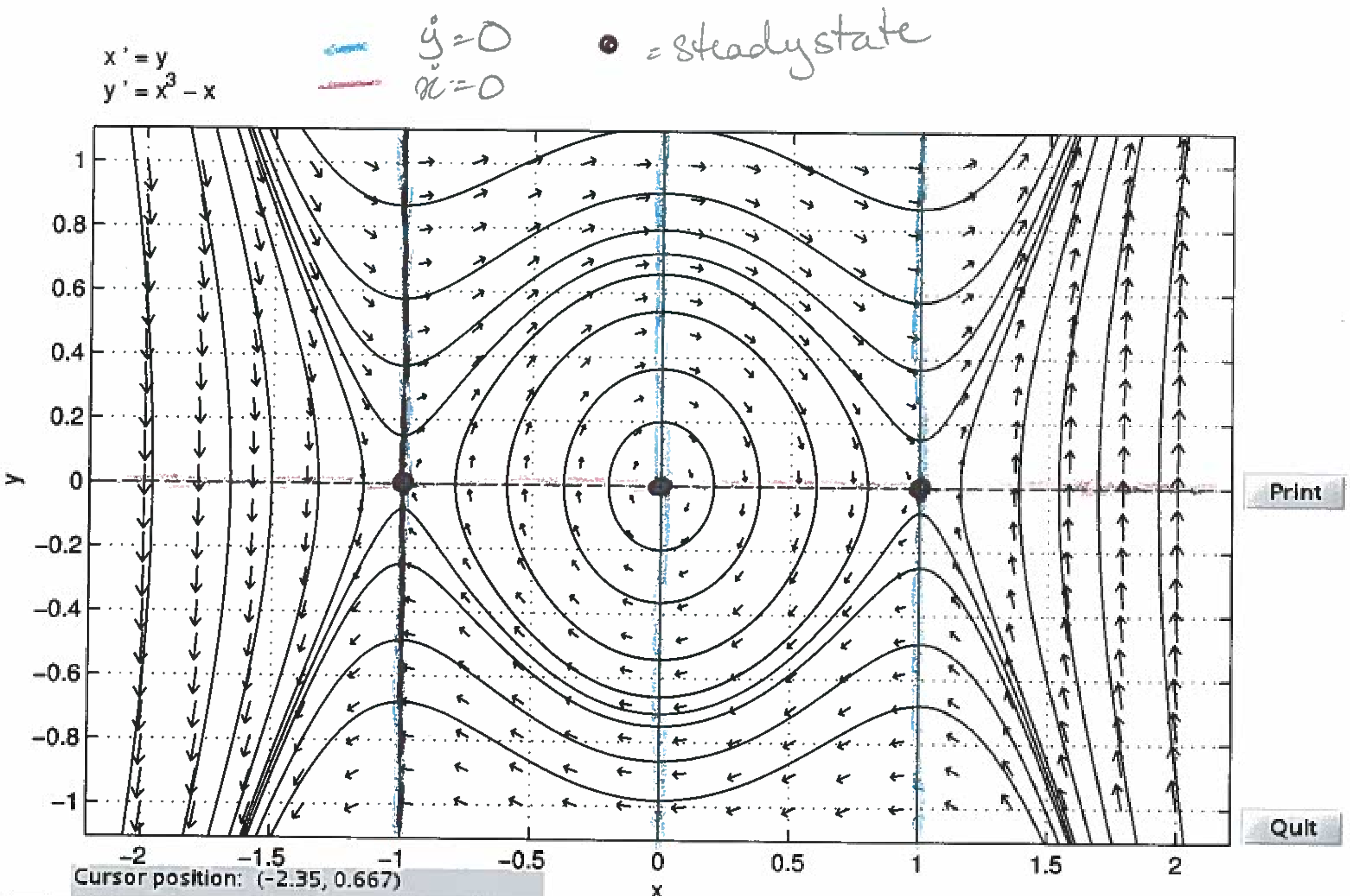
$$= \frac{1}{2}y^2 - \frac{x^4}{4} + \frac{x^2}{2} = E(x,y)$$

Level curves of E:



phase plane on next p.

Phase plane



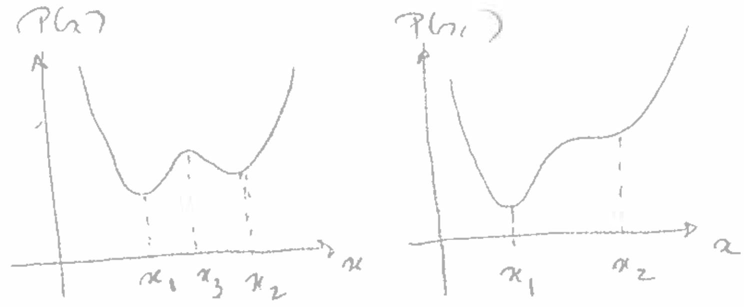
The backward orbit from (-2, 0.15) left the computation window.
Ready.
The forward orbit from (1.9, 0.096) left the computation window.
The backward orbit from (1.9, 0.096) left the computation window.
Ready.

c) irrelevant, as we did not do this problem in class!

2. $\ddot{x} + b\dot{x} + P'(x) = 0$

Normal Form:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -by - P'(x) \end{cases}$$

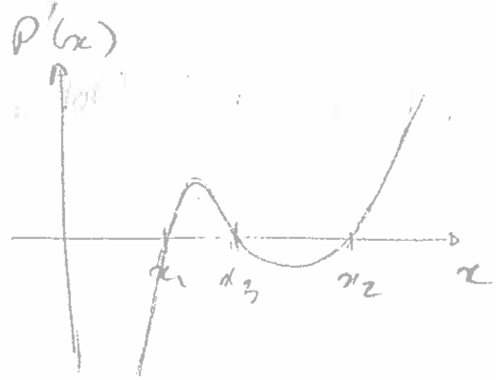


(a)

Steady states + nullclines

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ by + P'(x) = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ y = -\dfrac{P'(x)}{b} \end{cases}$$

(a)



Equilibria:

$(x_1, 0)$; $(x_2, 0)$;

$(x_3, 0)$

Stability

$$J = \begin{bmatrix} 0 & 1 \\ -P''(x) & -b \end{bmatrix}$$

$$\therefore |J-\lambda I| = 0 \iff -\lambda(-b-\lambda) + P''(x) = 0$$

$$\iff \lambda^2 + b\lambda + P''(x) = 0$$

$$\iff \lambda = \frac{-b \pm \sqrt{b^2 - 4P''(x)}}{2}$$

<u>At $(x_3, 0)$</u>

$P''(x_3) < 0$ $\therefore$ λ_+ & λ_- have opposite sign & so the steady state is a <u>saddle</u>.

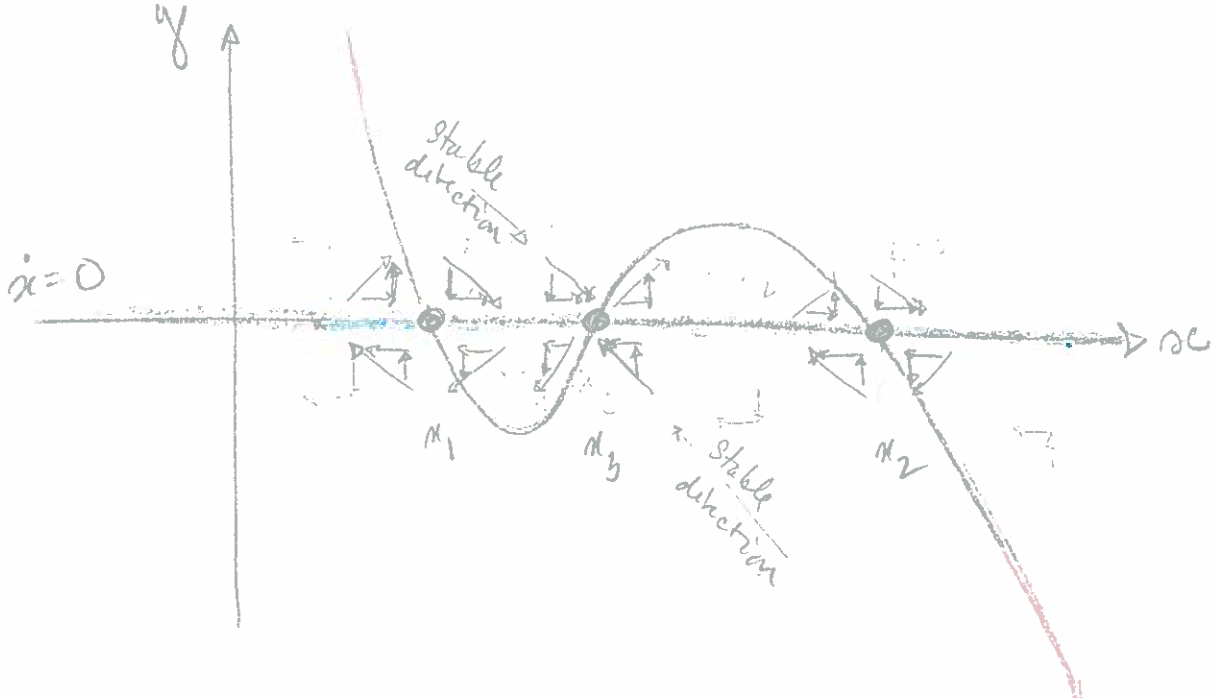
At $(x_1, 0)$ and $(x_2, 0)$

$P''(x) > 0$ $\therefore$ $\text{sgn}(\text{Re}(\lambda\pm)) = \text{sgn}(-b) < 0$.

$\therefore$ these steady states are <u>stable</u>. They are stable <u>nodes</u> for $b^2 > 4P''(x)$, & stable <u>foci</u> for $0 < b^2 < 4P''(x)$.

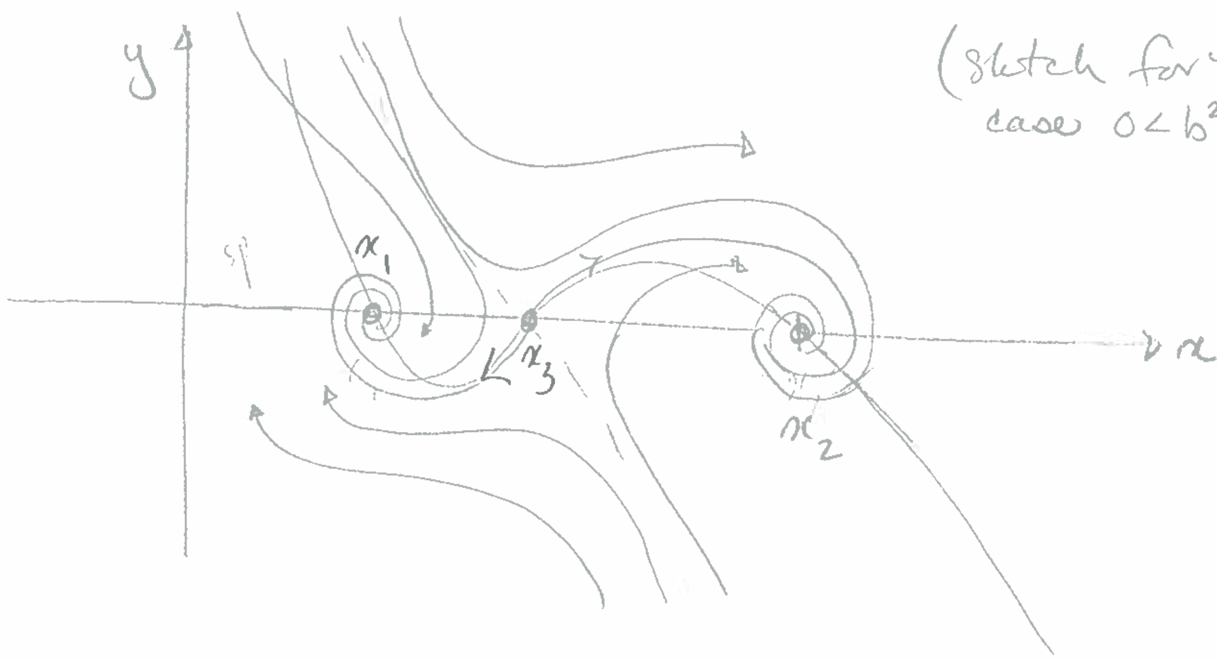
Phase plane:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \iff \begin{cases} y = 0 \\ y = -\dfrac{P'(x)}{b} \end{cases}$$

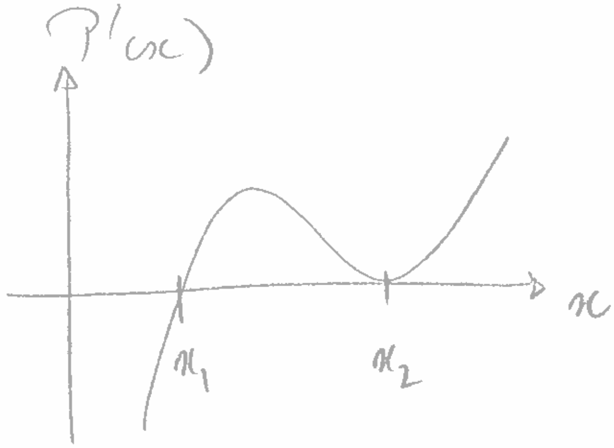


$\therefore$ we expect trajectories to look like:

(sketch for the case $0 < b^2 < 4P'(x)$)



(b)



Equilibria: $(x_1, 0)$; $(x_2, 0)$

Stability: $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4P''(x)}}{2}$

<u>At $(x_1, 0)$</u>

$P''(x_1) > 0$ and so $\mathrm{sgn}(\mathrm{Re}(\lambda)) = \mathrm{sgn}(-b) < 0$. This point is thus a <u>stable node</u> for $b^2 > 4P''(x)$ and a <u>stable focus</u> for $0 < b^2 < 4P''(x)$.

<u>At $(x_2, 0)$</u>

$P''(x) = 0$ and so $\lambda_+ = 0$, $\lambda_- = -b$. The linear analysis is thus inconclusive. So we use the energy function to determine stability.

$$E = \frac{1}{2}\dot{x}^2 + P(x) = \frac{1}{2}y^2 + P(x)$$

$$\dot{E} = y\dot{y} + P'(x)\dot{x} = y(-by - P'(x)) + P'(x)y$$

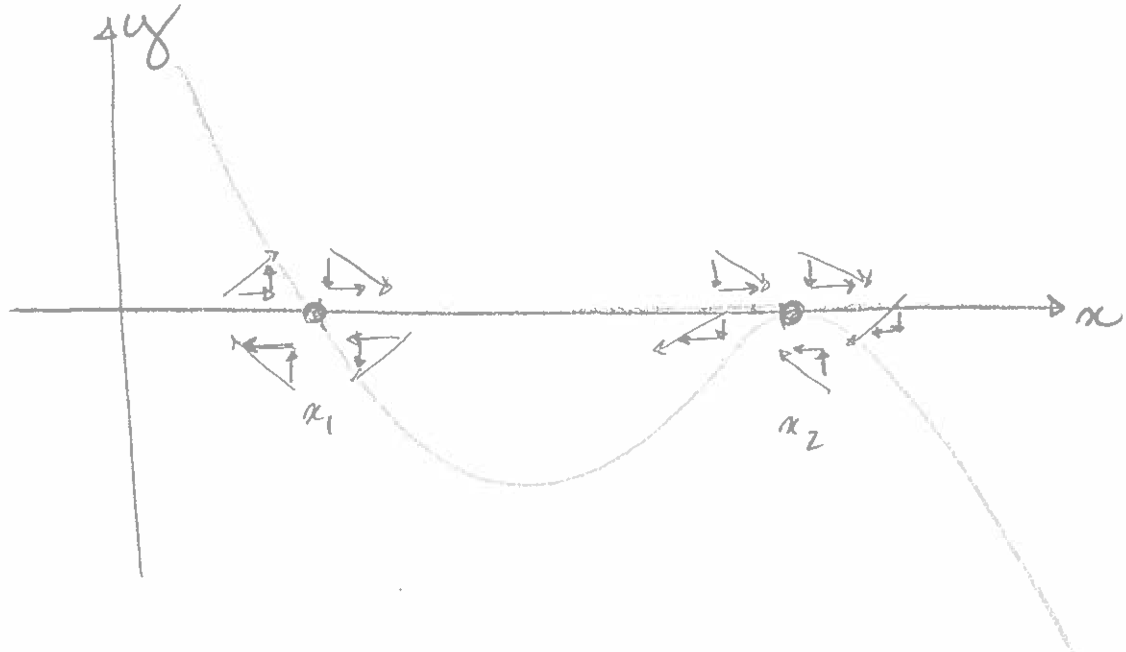
$$= -by^2 < 0$$

$$\therefore \dot{E} < 0 \ \forall y \neq 0$$

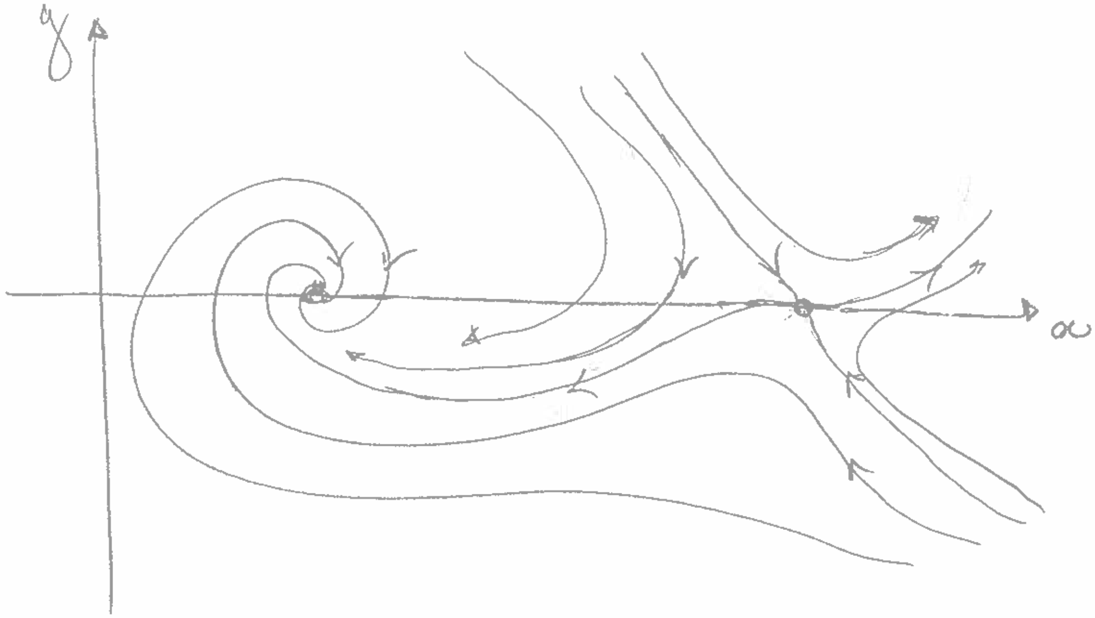
However, E is not a Lyapunov function for the steady state $(x_2, 0)$, because there is no neighbourhood of $(x_2, 0)$ in which $E(\vec{v}) > E(x_2, 0) \ \forall \ \vec{v}$ in the neighbourhood, $\because$ E increases monotonically for all intervals $[x_2 - \varepsilon, x_2 + \varepsilon]$.

Thus, the steady state $(x_2, 0)$ is not stable.

Phase plane



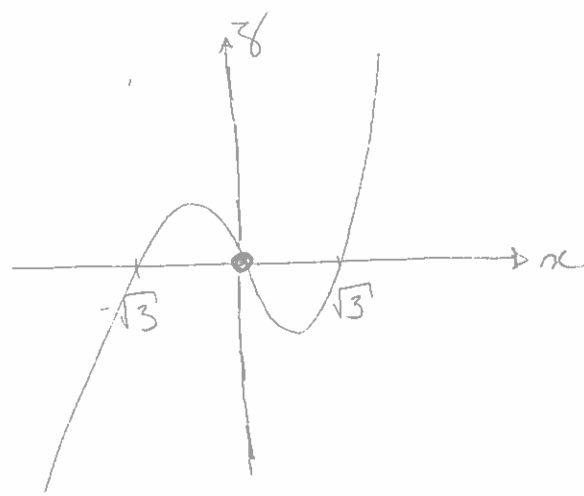
∴ we expect trajectories to look like:



3. $$\begin{cases} \dot{x} = -y + \left(\frac{x^3}{3} - x\right) \\ \dot{y} = x \end{cases}$$

a) Steadystates

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Leftrightarrow \begin{cases} -y + \frac{x^3}{3} - x = 0 \\ x = 0 \end{cases} \Leftrightarrow \begin{cases} y = x\left(\frac{x^2}{3} - 1\right) \\ x = 0 \end{cases}$$



Steadystates at $(0,0)$; $(\pm\sqrt{3}, 0)$

b) $V(x,y) = x^2 + y^2$

$V(0,0) = 0$; $V(\pm\sqrt{3}, 0) = 3$

$V(x,y) > 0 \ \forall \ (x,y) \neq (0,0)$

$V(x,y)$ can't be a Lyapunov fn for these steadystates

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial z}\dot{z} = 2x\dot{x} + 2z\dot{z}$$

$$= 2x\left(-z + x\left(\frac{x^2}{3} - 1\right)\right) + 2zx$$

$$= 2x^2\left(\frac{x^2}{3} - 1\right)$$

$$\therefore \dot{V} < 0 \text{ if } \frac{x^2}{3} - 1 \quad (\Leftrightarrow) \; -\sqrt{3} < x < \sqrt{3}$$

Thus $V(x,y)$ is a Lyapunov function for the steady state at $(0,0)$ in the region $-\sqrt{3} < x < \sqrt{3}$.

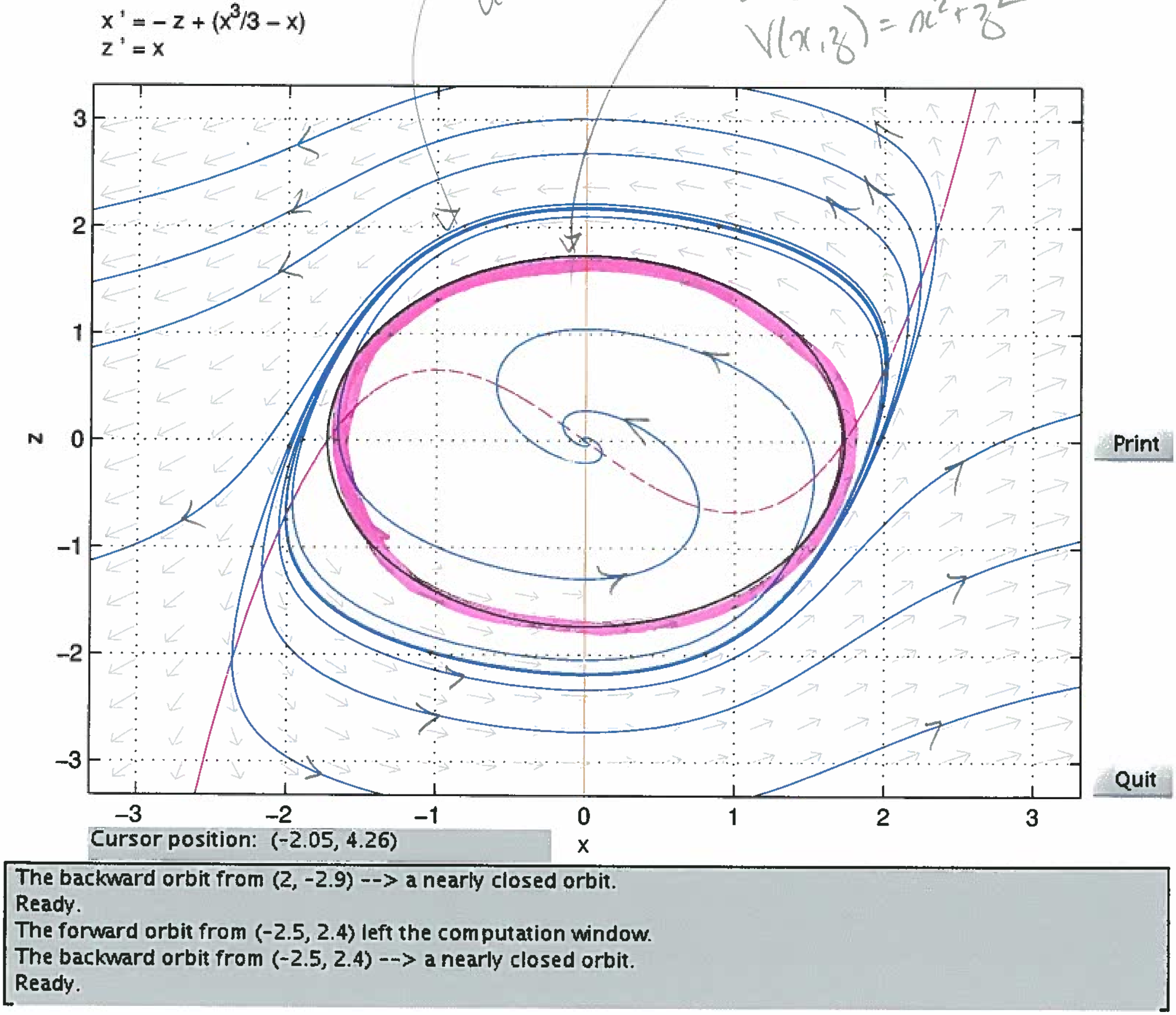
c) The basin of attraction is $\sqrt{3} < x < \sqrt{3}$.

d) All solutions with starting point $|x_0| < \sqrt{3}$ end up at $(0,0)$. All other points must either go to $\pm\infty$ or circle around the basin $-\sqrt{3} < x < \sqrt{3}$ $\forall t > 0$.

e) see next page

unstable limit cycle

basin of attraction as determined from $V(x,z) = x^2 + z^2$



We see that the trajectories inside the circle $x^2+z^2=3$ go to the steady-state at $(0,0)$, confirming that all points inside the circle are in the basin of attraction for $(0,0)$.

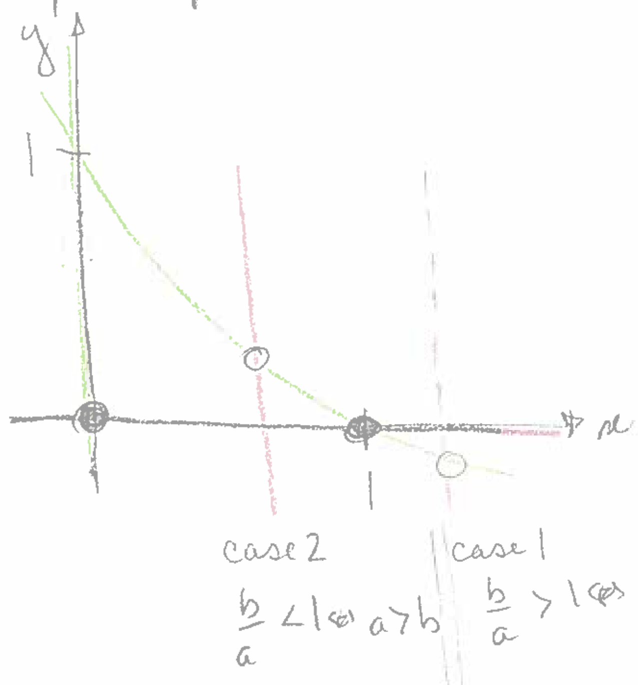
4. $\begin{cases} \dot{x} = x(1-x) - 2xy(1+x) \\ \dot{y} = axy - by \end{cases}$ $\quad a>0,\ b>0$

a) nullclines & steady states

$\begin{cases} \dot{x}=0 \\ \dot{y}=0 \end{cases} \Leftrightarrow \begin{cases} x(1-x) - 2xy(1+x) = 0 \\ axy - by = 0 \end{cases}$

$\Leftrightarrow \begin{cases} x = 0 \quad \text{OR} \quad 1-x-2y(1+x) = 0 \Leftrightarrow y = \left(\frac{1-x}{1+x}\right)\frac{1}{2} \\ y = 0 \quad \text{OR} \quad ax - b = 0 \quad \Leftrightarrow x = \frac{b}{a} \end{cases}$

$\therefore$ the phase plane is



Steady states at: $(0,0)$, $(1,0)$, & $\left(\frac{b}{a}, \frac{1}{2}\left(\frac{a-b}{a+b}\right)\right)$

The coexistence state only exists if $a > b$.

b) Stability

$$J = \begin{bmatrix} 1-2x-2y-4xy & -2x(1+x) \\ ay & ax-b \end{bmatrix}$$

$$J\Big|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -b \end{bmatrix}$$ $\therefore \lambda_1 = 1,\ \lambda_2 = -b$, and the origin is a saddle.

$$J\Big|_{(1,0)} = \begin{bmatrix} 1-2 & -2(2) \\ 0 & a-b \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 0 & a-b \end{bmatrix}$$

$\therefore \lambda_1 = -1,\ \lambda_2 = a-b$, and the prey only steady state is

- stable node if $a < b$ (coexistence state does not exist)
- saddle if $a > b$ (coexistence state does exist)

$$J\Big|_{\left(\frac{b}{a},\ \frac{a-b}{a+b}\frac{1}{2}\right)} = \begin{bmatrix} 1-\frac{2b}{a}-\frac{2(a-b)}{2(a+b)}-\frac{4}{2}\frac{b}{a}\frac{a-b}{a+b} & -2\frac{b}{a}\left(1+\frac{b}{a}\right) \\ \frac{a}{2}\left(\frac{a-b}{a+b}\right) & a\frac{b}{a}-b \end{bmatrix}$$

$$= \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = 1 - \frac{2b}{a} - \frac{2(a-b)}{2a+b} - \frac{4b}{2a}\,\frac{a-b}{a+b}$$

$$= 1 - \frac{2b}{a} - \frac{(a-b)}{a+b} - \frac{2b(a-b)}{a(a+b)}$$

$$= \frac{a(a+b) - 2b(a+b) - a(a-b) - 2b(a-b)}{a(a+b)}$$

$$= \frac{\cancel{a^2} + \cancel{ab} - \cancel{2ab} - \cancel{2b^2} - \cancel{a^2} + \cancel{ab} - 2ab + \cancel{2b^2}}{a(a+b)}$$

$$= -\frac{2b}{a+b}$$

$$J_{12} = -\frac{2b(a+b)}{a^2} \qquad J_{21} = \frac{a}{2}\left(\frac{a-b}{a+b}\right)$$

$$J_{22} = 0$$

Eigenvalues:

$$|J - \lambda I| = 0 \iff \left(\frac{-2b}{a+b} - \lambda\right)(-\lambda) + \frac{\cancel{2}b\cancel{(a+b)}}{a^{\cancel{2}}}\,\frac{\cancel{a}}{\cancel{2}}\left(\frac{a-b}{\cancel{a+b}}\right) = 0$$

$$\iff \lambda^2 + \frac{2b}{a+b}\lambda + \frac{b(a-b)}{a} = 0$$

$$\iff a(a+b)\lambda^2 + 2ab\lambda + b(a^2-b^2) = 0$$

$$\iff \lambda = \frac{-ab \pm \sqrt{(ab)^2 - (a+b)b(a^2-b^2)}}{a(a+b)}$$

Let $A = \dfrac{-b}{a+b}$, $B = (a+b)(a^2-b^2)b$

We know that $B>0$ in the regime where the coexistence state exists. There are thus three cases:

i) $\underline{b^2 - B > 0}$

Then $\lambda_\pm$ have the same sign as A.

(ii) $\underline{b^2 - B = 0}$

Then $\lambda_\pm = \lambda$ + has the same sign as A.

iii) $\underline{b^2 - B < 0}$

Then $\lambda_\pm$ complex & $Re(\lambda_\pm)$ has the same sign as A.

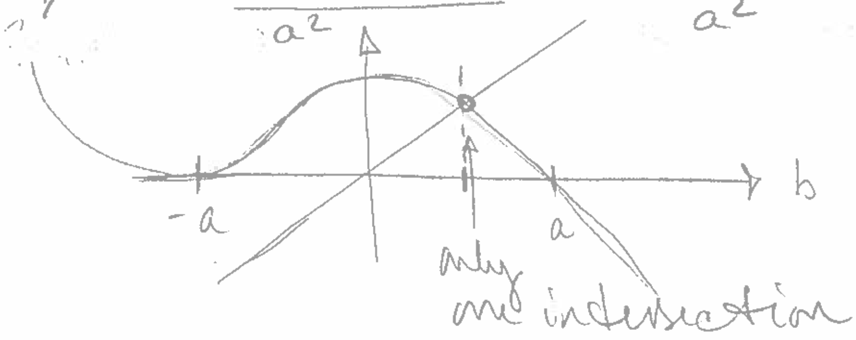
$\therefore$ The sign of A determines the stability of the steady state. But $\mathrm{sgn}(A) = \mathrm{sgn}(-b)$, and so

$$Re(\lambda_\pm) < 0$$

and the coexistence state is always $\underline{\underline{\text{stable}}}$, either a node or a focus. The boundary between node & focus behaviour occurs at

$$(ab)^2 - (a+b)(a^2-b)^2 b = 0 \Leftrightarrow b = 0 \text{ or } ab - (a+b)(a^2-b^2) = 0 \Leftrightarrow$$

$$\Leftrightarrow b = \frac{(a+b)(a^2-b^2)}{a^2} = \frac{(a+b)^2(a-b)}{a^2}$$



Let b^* be the value of b at the intersection point.
For $b > b^*$, we have

$$b > \frac{(a+b)(a^2-b^2)}{a^2} \Leftrightarrow b^2 - \frac{(a+b)(a^2-b^2)}{a^2} b > 0$$

and so $\lambda_\pm$ are both real, & the steady state is a stable <u>node</u>. For $b < b^*$, the steady state is a stable <u>focus</u>.

Summary

- $(0,0)$ is a saddle
- $(1,0)$ is a stable <u>node</u> for $b > a$

  saddle for $b < a$

- $\left(\frac{b}{a}, \frac{1}{2}\frac{(a-b)}{(a+b)}\right)$ is a stable <u>node</u> for $b^* < b < a$

  stable <u>focus</u> for $0 < b < b^*$

where $b^* = \frac{(a+b^*)^2(a-b^*)}{a^2}$.

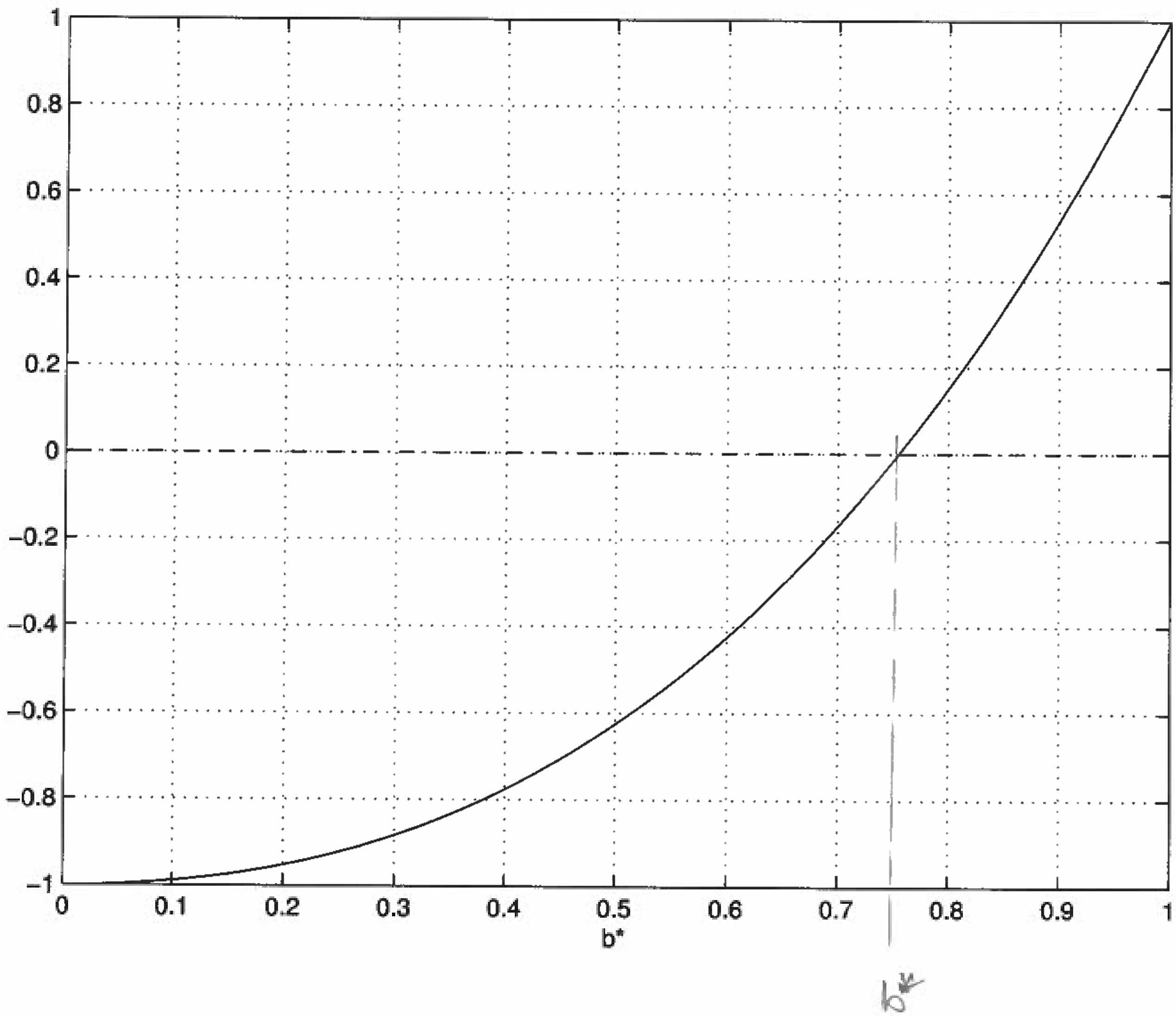
c) If $a=1$ then b^* is given by

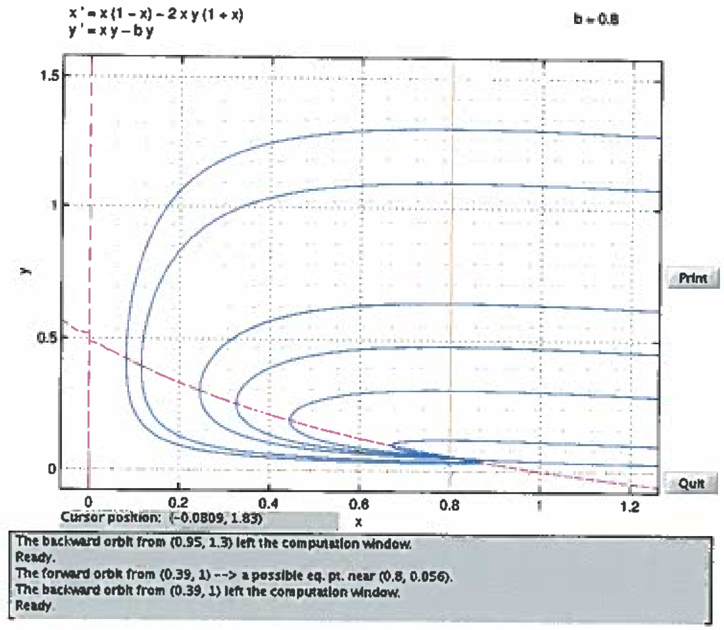
$$b^* = (1+b^*)^2(1-b^*) \Leftrightarrow b^* = (1+b^*)(1-b^{*2}) \Leftrightarrow$$

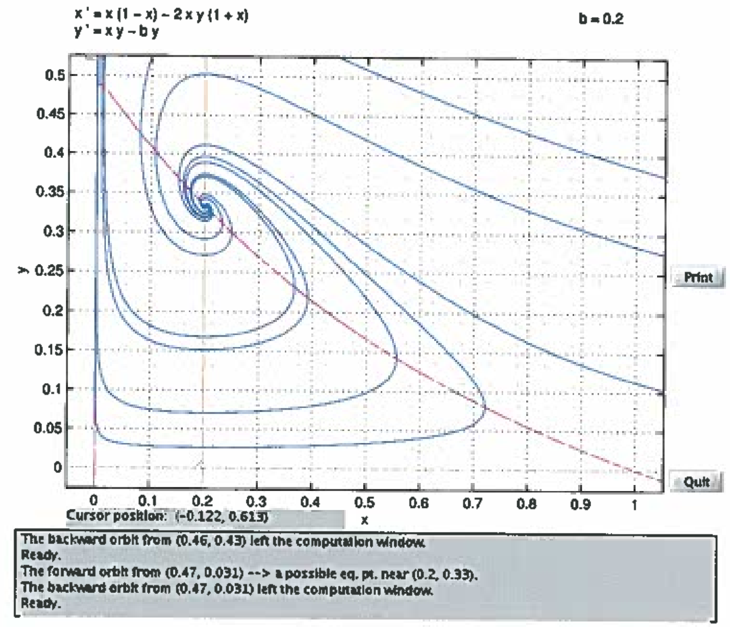
$$\Leftrightarrow b^* = 1 - b^{*2} + b^* - b^{*3} \Leftrightarrow b^{*3} + b^{*2} - 1 = 0$$

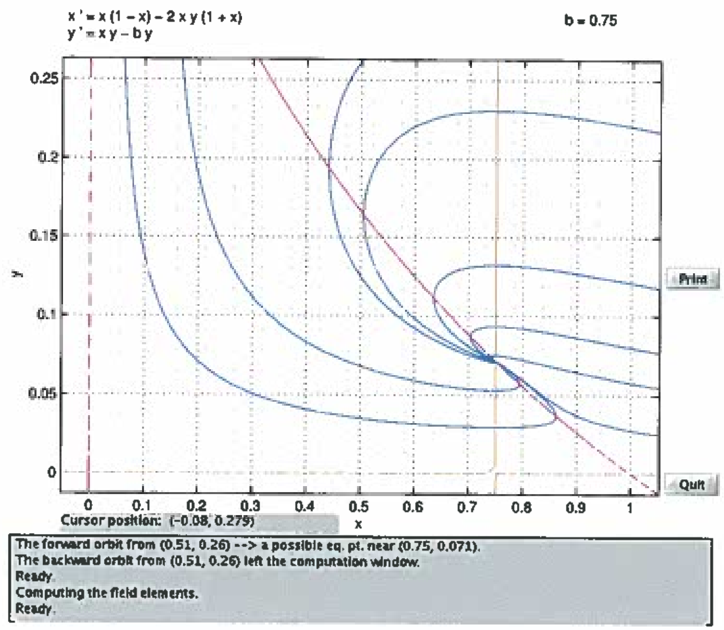
This is a cubic. We can find the solution numerically.
We obtain $b^* \simeq 0{,}75$ (see p 20)

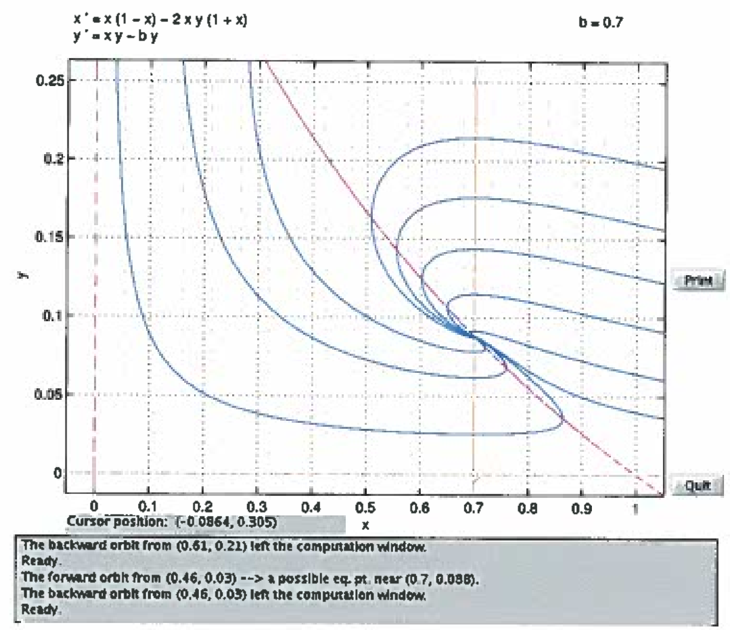
d) see p 21.


1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
b*
b*









Stable node

stable focus

(the difference is hard to
see in the bottom two plots)