INSTRUCTOR: Rebecca Tyson

COURSE: MATH 339

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Date: Oct 2nd, 2015 Time: 12:00pm Duration 45 minutes.
This exam has 4 questions for a total of 23 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Problem	Points Earned	Points Out Of
1		5
2		5
3		4
4		3
5		5
TOTAL:		**23**

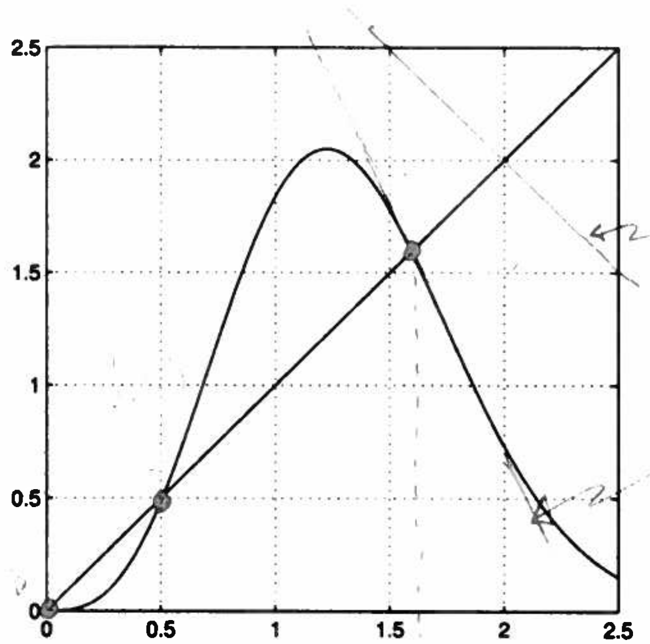
CANDIDATE NAME (print): Solutions

STUDENT NUMBER: __________

Signature: __________

w/ the identity line

$\boxed{5}$ 1. Consider the map $f(x)$ plotted in Figure 1. Indicate all of the steady states on the plot, and then determine the stability of each steady state. Explain how you arrive at your stability results.



this is what a slope of -1 looks like, so this slope is steeper

Figure 1: Map for problem 1.

By comparing the slope of $f(x)$ with the identity line, we see that

- at $x^* = 0$, $f'(x^*) = 0$, $\therefore$ the steady state is stable
- at $x^* = \frac{1}{2}$, $f'(x^*) > 1$, $\therefore$ " " " unstable

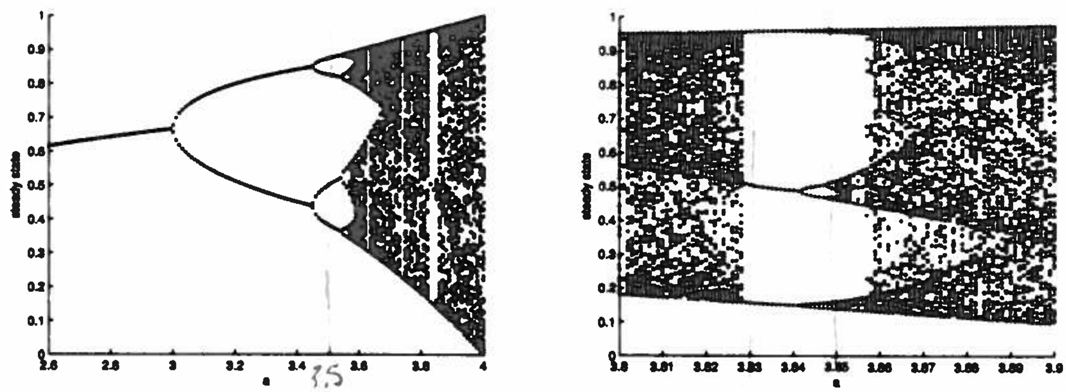
At $x^* \simeq 1.6$, we see that the line tangent to $f(x)$ has a negative slope that is steeper than -1 (see lines drawn), and so

- at $x^* \simeq 1.6$, $|f'(x)| > 1$, $\therefore$ the steady state is unstable

(Note: cobwebbing can also be used)

5 2. Below are two plots showing parts of the bifurcation diagram for the logistic map, $f(x) = ax(1-x)$. The right panel is a zoomed-in view of part of the left panel.

(a) For each range of a values given below, describe what is the behaviour of the system (i.e. what is the period of the stable orbit, any other special names for the behaviour or sequence of behavours).



1 i. $2.6 < a < 3.5$

Here we see a period-doubling cascade. The stable orbit is initially 1 point, then a period-2 orbit, then a period-4 orbit.

1 ii. $3.83 < a < 3.85$

Here we see another period-doubling cascade, starting from a period-3 orbit, then becoming a period-6, followed by a period-12

1 iii. $a = 4$

chaos (orbit has no periodicity + has sensitive dependence on initial conditions)

2 (b) At $a = 3$, the left panel shows a bifurcation in model behaviour. Mathematically, what ~~happens to~~ can you say about the stability of the fixed point x^* of f ~~as a increases past 3~~ when $a<3$, $a=3$, + $a>3$? Each point in the period-2 orbit ~~that~~ $\{p_1, p_2\}$ emerges at $a>3$ is a fixed point of what map?

for $a < 3$, $|f'(x^*)| < 1$, at $a = 3$ $|f'(x^*)| = 1$, for $a > 3$, $|f'(x^*)| > 1$; the period-2 orbit points p_1 + p_2 are stable fixed points of the f^2 map.

$\boxed{4}$ 3. Consider the linear map $f(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1/4 & -1/2 \\ 1/2 & 3/4 \end{bmatrix}.$$

Determine the stability of the steady state at the origin.

eigenvalues.

$$|A - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} \frac{1}{4} - \lambda & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{4} - \lambda \end{vmatrix} = 0 \Leftrightarrow \left(\frac{1}{4} - \lambda\right)\left(\frac{3}{4} - \lambda\right) + \frac{1}{4} = 0$$

$$\Leftrightarrow \lambda^2 - \lambda + \frac{3}{16} + \frac{1}{4} = 0 \Leftrightarrow \lambda^2 - \lambda + \frac{3+4}{16} = 0$$

$$\Leftrightarrow \lambda^2 - \lambda + \frac{7}{16} = 0 \Leftrightarrow \lambda = \frac{1 \pm \sqrt{1 - \frac{7}{4}}}{2}$$

$$\Leftrightarrow \lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{-3}{4}} = \frac{1}{2} \pm \frac{\sqrt{3}}{4} i$$

$$|\lambda_i| = \sqrt{\left(\frac{1}{2}\right)^2 + \frac{3}{16}} = \sqrt{\frac{1}{4} + \frac{3}{16}} = \sqrt{\frac{7}{16}} < 1$$

$\therefore$ the steady state is a sink.

3 $\cancel{\boxed{4}}$ 4. Below are three plots showing the image of the unit circle under two iterations of three different maps. For each plot, classify the steady state at the origin. Explain your answer in each case.

Image 1

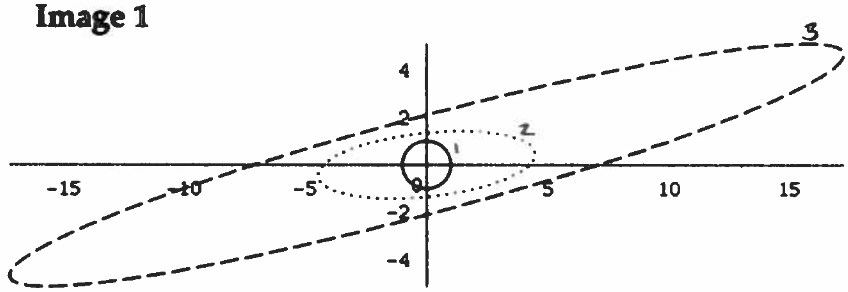


Image 2

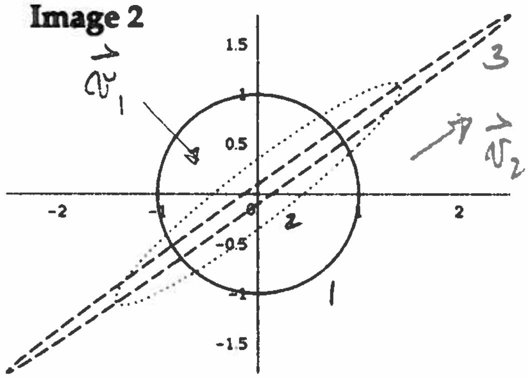


Image 3

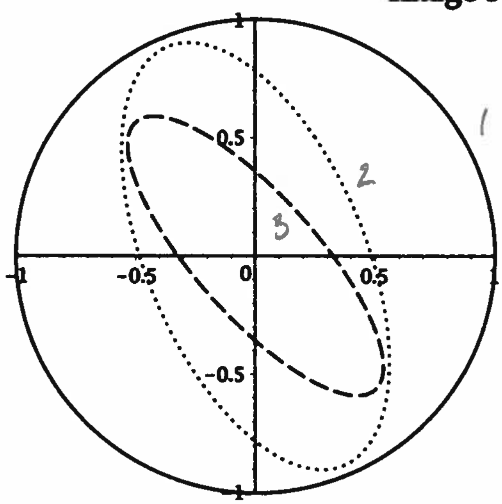


Image 1: The steady state is a source because each image is larger than the one before in all directions.

Image 2: The steady state is a saddle because each subsequent image is smaller than the previous one in one direction ($\vec{v}_1$) and larger in the other ($\vec{v}_2$).

Image 3: The steady state is a sink because each image is smaller than the one before in all directions.

5. Consider the saddle fixed point $\vec{0}$ of the map $f(x, y) = (x/2, 2y - 7x^2)$, and the following information:

   - The inverse map is $f^{-1}(x, y) = (2x, y/2 + 14x^2)$.
   - The y axis is invariant under f, and $|f^{-n}(0, y) \to 0|$ as $n \to \infty$.

   With this information, answer the questions below.

$\boxed{3}$ (a) Show that the set $S = \{(x, 4x^2) : x \in \mathbb{R}\}$ is invariant under f, and that $|f^n(x, 4x^2) \to 0|$ as $n \to \infty$.

$f(x, 4x^2) = \left(\frac{x}{2}, 2(4x^2) - 7x^2\right) = \left(\frac{x}{2}, 8x^2 - 7x^2\right)$
$= \left(\frac{x}{2}, x^2\right) = \left(\frac{x}{2}, 4\left(\frac{x}{2}\right)^2\right) \to$ still on the curve $y = 4x^2$

$\therefore$ the set S is invariant under f. Also, $\because$ x maps to $\frac{x}{2}$, and the curve $y = 4x^2$ goes through the origin, $f^n(x, 4x^2) \to 0$ as $n \to \infty$.

$\boxed{2}$ (b) Sketch the stable and unstable manifolds of $\vec{0}$, and some nearby trajectories.