Math 339 - Fall 2017
Assignment #1 - Sol'ns



1) Exercise 1.2 p 36

a) $f(x) = x - x^2$

$\therefore \; x_{n+1} = x_n - x_n^2$

• fixed pts: $x_{n+1} = x_n = x^* \iff x^* = x^* - x^{*2}$

$\therefore$ either $x^* = 0$ or $1 = -x^{*2}$, which has no real solution

So $x^* = 0$ is the only fixed point.

• stability:

$$\left| f'(x) \Big|_{x^*} \right| = \left| 1 - 2x^* \right| = |1| = 1$$

This test is inconclusive. So we use cobwebbing.

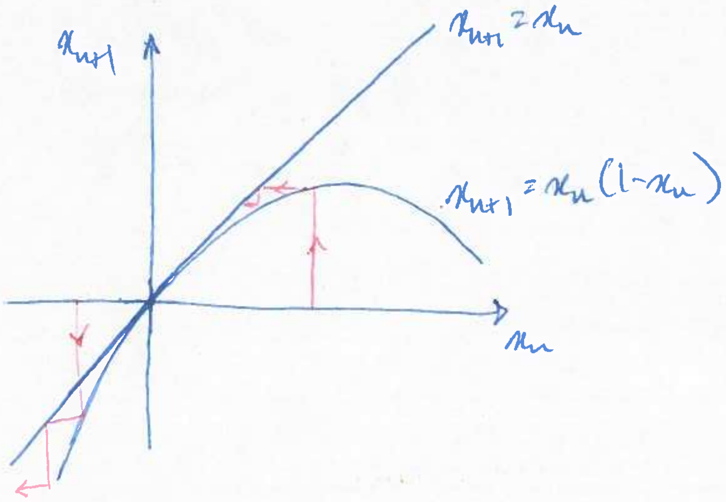



Fig 1.

From Fig 1, We see that $x_n(1-x_n) < x_n$ $\forall x_n$. $\therefore$ for $x_n > 0$, the origin is a sink, & for $x_n < 0$, the origin is a source.

b) Let $g(x) = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$.

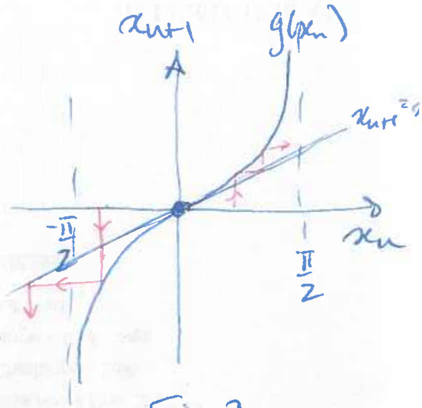



Fig 2.

- fixed points

$$x^* = g(x_n^*) \iff x^* = \tan(x^*)$$

$$\iff x^* = 0$$

- Stability:

$$\left| g'(x) \right|_{x^*} = \left| \frac{1}{\cos^2(x^*)} \right| = |1| = 1$$

Again, the test is inconclusive, so we use cobwebbing.

In this case, $\begin{cases} g(x_n) > x_n \ \forall\, x_n > 0, \\ g(x_n) < x_n \ \forall\, x_n < 0. \end{cases}$

Thus, by Fig 2. we see that the origin is a <u>source</u>.

c)

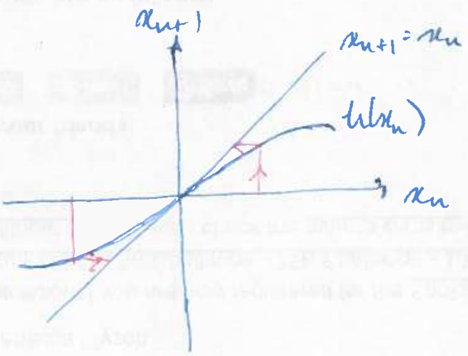



Fig 3.

Any function $h(x)$ which satisfies $h'(0) = 1$ and

$$\begin{cases} h(x_n) < x_n & \forall\, x_n > 0 \\ h(x_n) > x_n & \forall\, x_n < 0 \end{cases}$$

will have a sink at $x = 0$.

Example: $h(x) = \sin(x)$

2) Exercise 1.5 p36

Let $f(x) = 2x^2 - 5x = x(2x-5)$.

So $x_{n+1} = x_n(2x_n - 5)$

$x_{n+2} = x_{n+1}(2x_{n+1} - 5) = x_n(2x_n-5)(2x_n(2x_n-5)-5)$.

- fixed pts:
  - of $f(x)$ are
    $x = x(2x-5) \iff x^* = 0$ or $1 = 2x-5 \iff x^* = 3$
  - of $f^2(x)$ are $x^* = 0$, $x^* = 3$, and
    $x = x(2x-5)(2x(2x-5)-5) \iff 1/$

$1/ \iff 1 = (2x-5)(4x^2-10x-5)$

$\iff 1 = 8x^3 - 20x^2 - 10x - 20x^2 + 50x + 25$

$\iff 8x^3 - 40x^2 + 40x + 24 = 0$

$\iff 2x^3 - 10x^2 + 10x + 6 = 0$

$\iff x^3 - 5x^2 + 5x + 3 = 0 \quad (1)$

We know that $x = 3$ must be a root, so we divide:

$$
\begin{array}{r|l}
 & x^2 - 2x - 1 \\
\hline
x-3 & x^3 - 5x^2 + 5x + 3 \\
 & x^3 - 3x^2 \\
\hline
 & -2x^2 + 5x \\
 & -2x^2 + 6x \\
\hline
 & -x + 3 \\
 & -x + 3 \\
\hline
 & 0
\end{array}
$$

$\therefore$ (1) becomes

$$(x-3)(x^2-2x-1)=0$$

So we have $x=3$ or

$$x^2-2x-1=0 \text{ so } x = 1 \pm \sqrt{1+1} = 1 \pm \sqrt{2}$$

$\therefore$ the period-2 orbit is

$$\begin{cases} x^*_n = 1+\sqrt{2} \\ x^*_{n+1} = 1-\sqrt{2} \end{cases}$$

- Stability

To determine if this orbit is a sink or a source we look at

$$\left| f'(x)\Big|_{x^*_n} f'(x)\Big|_{x^*_{n+1}} \right| = I$$

We find

$$f(x) = 2x^2-5x$$

$$f'(x) = 4x-5$$

and so

$$\left| f'(1+\sqrt{2})\, f'(1-\sqrt{2}) \right| = \left| \left(4(1+\sqrt{2})-5\right)\left(4(1-\sqrt{2})-5\right) \right| = I$$

$$I = \left| 16(1-2) - 20(1+\sqrt{2}) - 20(1-\sqrt{2}) + 25 \right|$$

$$= \left| -16 - 40 + 25 \right| = \left| -56 + 25 \right| = \left| -31 \right| = 31 > 1$$

$\therefore$ the period-2 orbit is a source.

3. $f(x) = r x^2 e^{-5x}$

a) steady state:

$$x^* = r x^{*2} e^{-5x^*} \iff x^* = 0 \quad \text{OR} \quad 1 = r x^* e^{-5x^*} \quad \ldots \quad (1)$$

the bifurcation occurs when

$$|f'(x^*)| = 1$$

From graphical exploration, we deduce, furthermore, that

$$f'(x^*) = -1 \iff 2r_c x^* e^{-5x^*} - 5r_c x^{*2} e^{-5x^*} = -1 \quad \ldots (2)$$

b) Using (1), (2) becomes (with $r = r_c$)

$$2 - 5x^* = -1 \iff x^* = \frac{3}{5} \quad \ldots\ldots \quad (3)$$

Plugging (3) into (2) we obtain

$$r_c \left(2x^* e^{-5x^*} - 5x^{*2} e^{-5x^*}\right) = -1 \iff$$

$$\iff r_c = \frac{e^{5x^*}}{2x^* - 5x^{*2}}$$

$$= \frac{e^3}{(2-3)\frac{3}{5}} = -\frac{5}{3}e^3$$

$$\simeq 33.5$$

c) See Maple output on following pages

```
> restart:
> with(plots);
```

[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, shadebetween, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot] **(1)**

```
> F := n → r·(n^2)·exp(-5·n);
```

$$F := n \to r\, n^2\, e^{-5\, n} \quad \textbf{(2)}$$

First, we let r take on a value less than r_c.

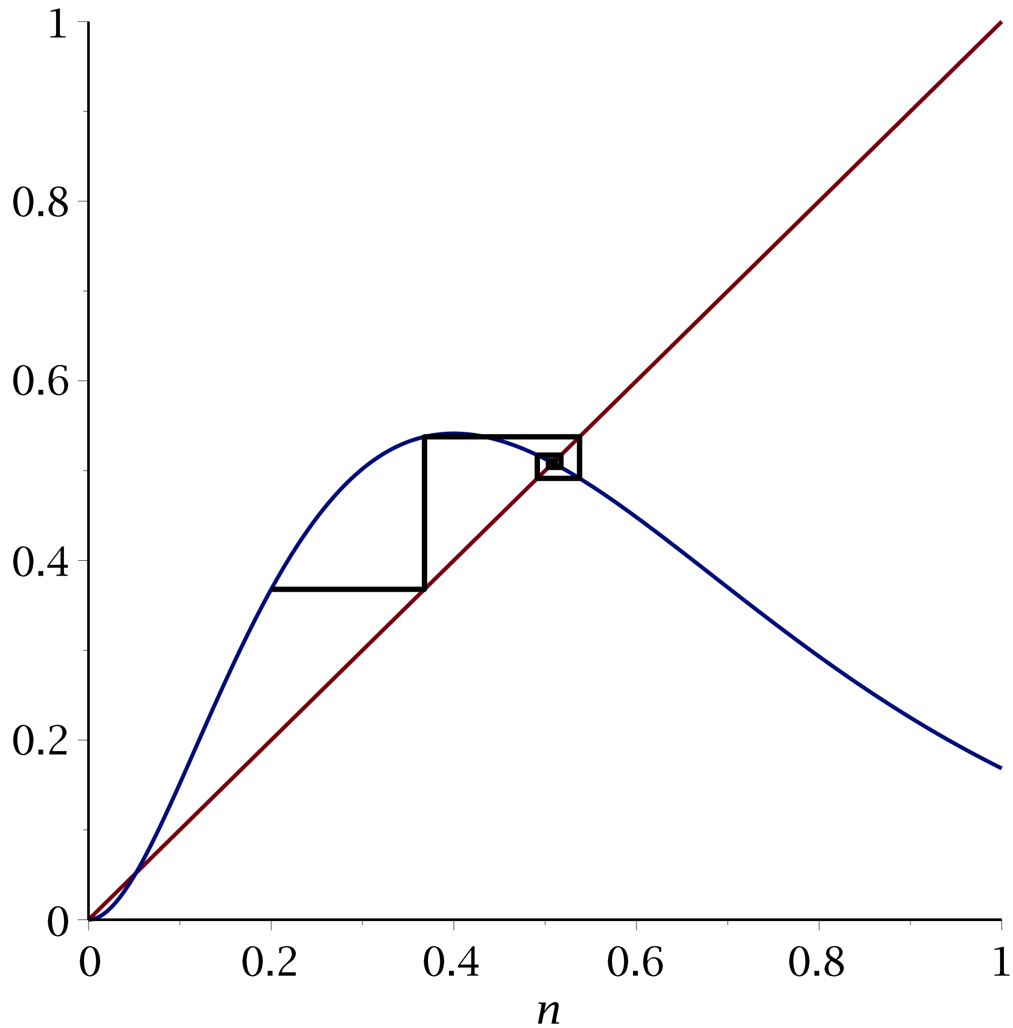
```
> n[0] := 0.2; r := 25; iterations := 10;
```

$$n_0 := 0.2$$
$$r := 25$$
$$iterations := 10 \quad \textbf{(3)}$$

```
> plotlist := NULL:
> x := n[0];
```

$$x := 0.2 \quad \textbf{(4)}$$

```
> for j from 1 to iterations do
   y := evalf(F(x)):
   plotlist := plotlist, [x, y], [y, y]:
   x := y;
   od:
> P := plot([plotlist], colour = black, thickness = 2):
> C := plot({F(n), n}, n = 0..1.0):
> display({P, C});
```




We see that in this case, the positive steady state is stable.

Now we look at r > r_c.

> $n[0] := 0.2;\ r := 40;\ iterations := 10;$

$$n_0 := 0.2$$

$$r := 40$$

$$iterations := 10 \qquad (5)$$

> $plotlist := NULL:$

> $x := n[0];$

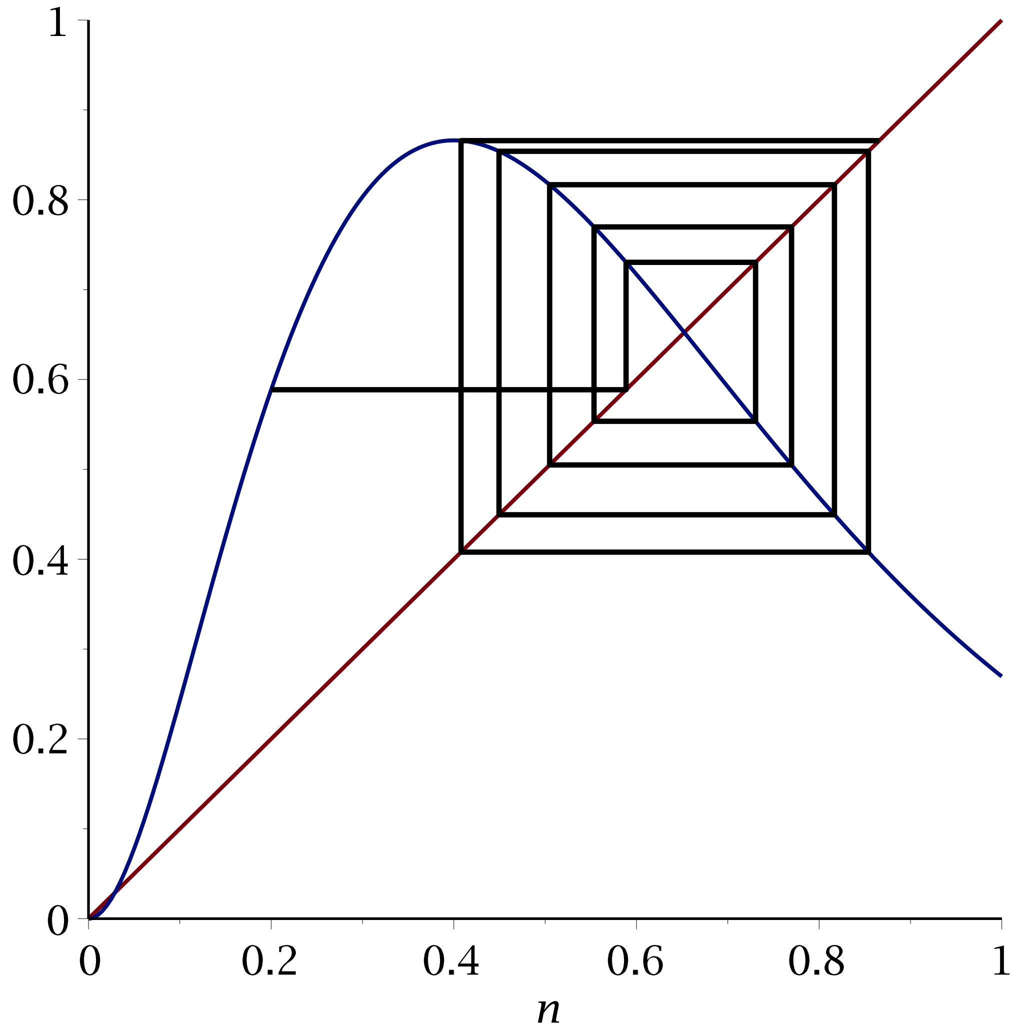
$$x := 0.2 \qquad (6)$$

> **for** $j$ **from** 1 **to** $iterations$ **do**
  $y := evalf(F(x)):$
  $plotlist := plotlist, [x, y], [y, y]:$
  $x := y;$
  **od**:

> $P := plot([plotlist], colour = black, thickness = 2):$

> $C := plot(\{F(n), n\}, n = 0..1.0)$ :

> $display(\{P, C\})$;




Here we see that the system is not attracted to the positive steady state. To see what the final behaviour is, we omit the transients from the plot.

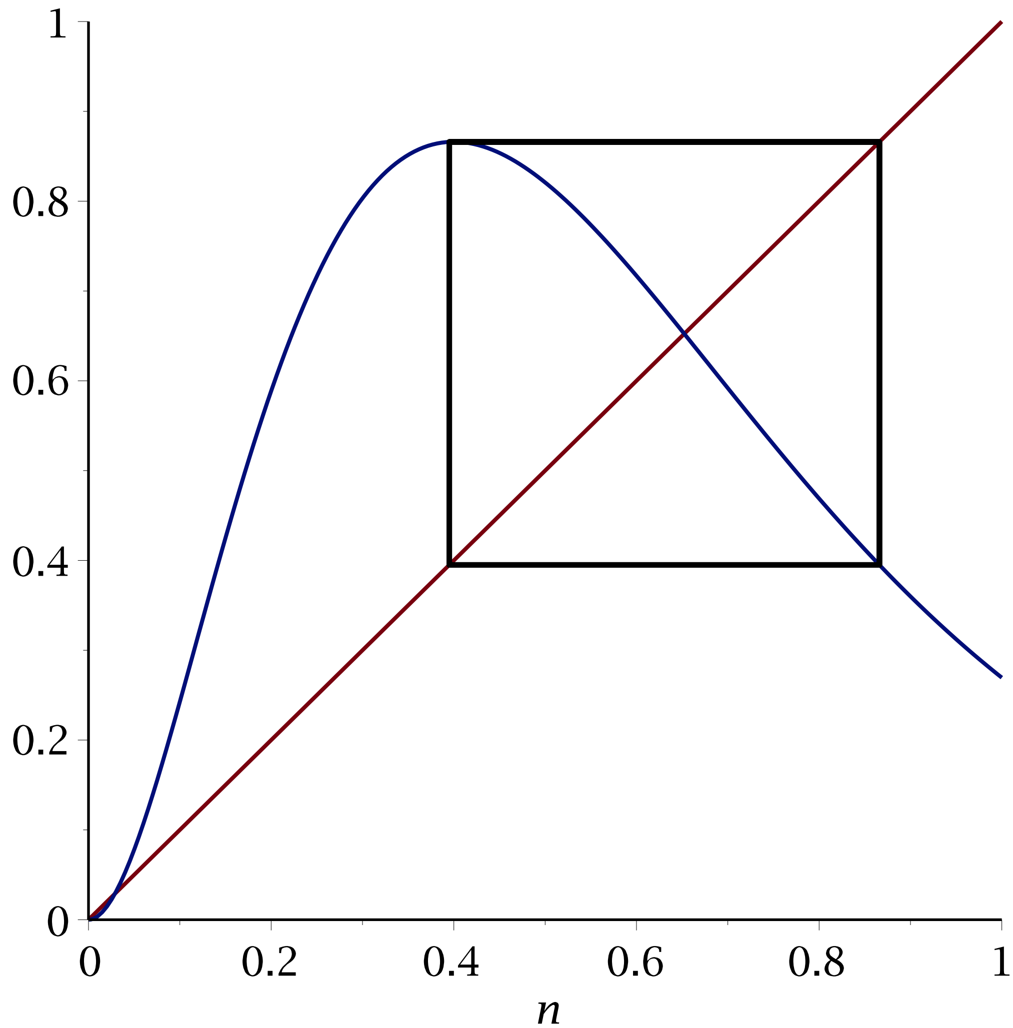
> $plotlist := NULL$ :

> **for** $j$ **from** 1 **to** $iterations$ **do**
>   $y := evalf(F(x))$ :
>   $plotlist := plotlist, [x, y], [y, y]$ :
>   $x := y$;
>   **od**:

> $P := plot([plotlist], colour = black, thickness = 2)$ :

> $C := plot(\{F(n), n\}, n = 0..1.0)$ :

> $display(\{P, C\})$;




We see that the system has a stable two-cycle. So the bifurcation at r=r_c is a period-doubling bifurcation.

We now plot the bifurcation diagram.

> $plotlist := NULL : n[0] := 1;\ transiterations := 100;$

$$n_0 := 1$$

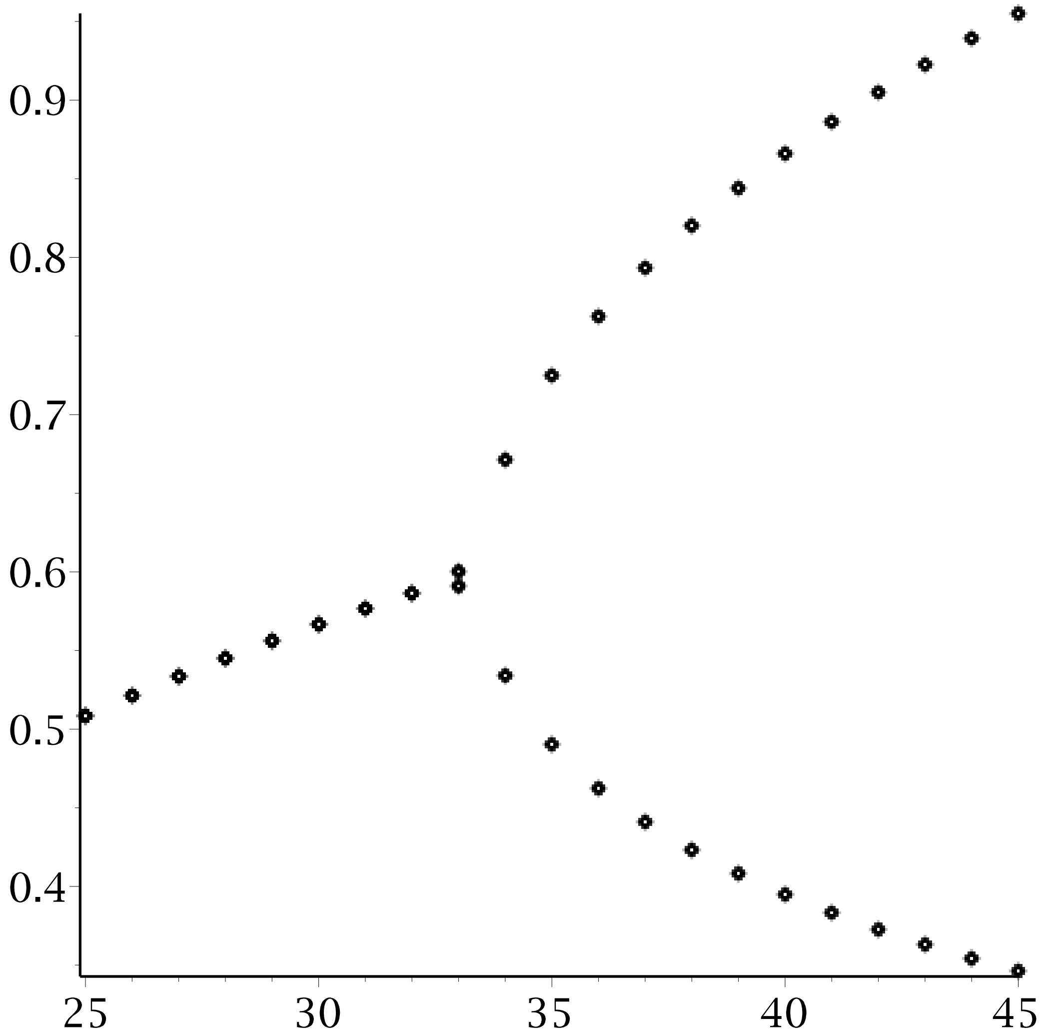
$$transiterations := 100 \quad \textbf{(7)}$$

```
> for r from 25 by 1 to 45 do
   x := n[0]:
   for j from 1 to transiterations do
   y := evalf(F(x)):
   x := y;
   od:
  for j from 1 to iterations do
   y := evalf(F(x)):
   plotlist := plotlist, [r, x]:
   x := y;
```

**od**:
**od**:

> $P := pointplot([plotlist], colour = black, thickness = 2)$ :

> $display(\{P\})$;




Here we see the bifurcation plot, showing a period-doubling bifurcation near the critical value r=r_c.