Assignment #2 - 2017
Solutions

1. (exercise 1.12)

$h(x) = axe^{-x}, \; a > 0$

a) fixed points:

$x^* = h(x^*) \iff x^* = ax^*e^{-x^*}$

$\iff x^* = 0$ or $1 = ae^{-x^*} \iff e^{-x^*} = \frac{1}{a}$

$\iff e^{x^*} = a$

$\iff x^* = \ln(a)$

b) stability:

$h'(x) = ae^{-x} - axe^{-x} = a(1-x)e^{-x}$

i) of $x^* = 0$

$h'(0) = a$

This steady state is stable if $0 < a < 1$.

ii) of $x^* = \ln(a)$

$h'(\ln(a)) = a(1-\ln(a))e^{-\ln(a)}$

$= a(1-\ln(a))\frac{1}{a} = 1 - \ln(a)$

$\therefore$ this steady state is stable if $0 < \ln(a) < 2$

and is superstable if $\ln(a) = 1 \iff a = e$

Continued on Maple output.

```
> restart:
> with(plots):
> F := n → a·n·exp(-n);
```

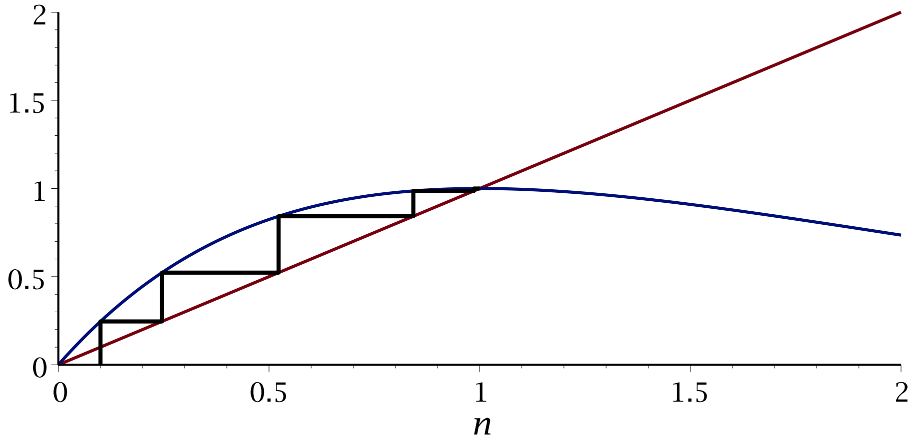
$$F := n \rightarrow a\, n\, e^{-n} \quad (1)$$

1(c) Consider the case where the nonzero steady state is superstable.

```
> n[0] := 0.1; a := exp(1); iterations := 10;
```

$$n_0 := 0.1$$
$$a := e$$
$$iterations := 10 \quad (2)$$

```
> plotlist := NULL:
> x := n[0]:
> plotlist := [x, 0]:
> for j from 1 to iterations do
   y := evalf(F(x)):
   plotlist := plotlist, [x, y], [y, y]:
   x := y;
  od:
> P := plot([plotlist], colour = black, thickness = 2):
> C := plot({F(n), n}, n = 0..2.0):
> display({P, C});
```



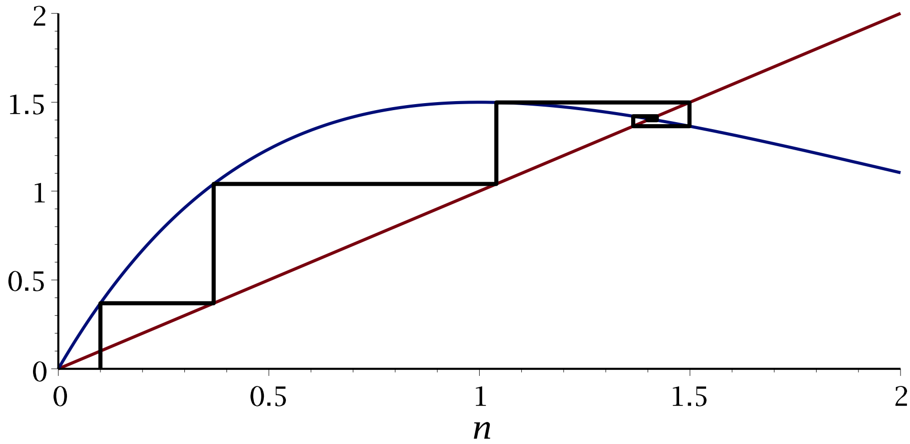
We see that the solution approaches the steady state monotonically.

Now increase a by 50%.

```
> a := 1.5·exp(1);
```

$$a := 1.5\, e \quad (3)$$

```
> plotlist := NULL:
> x := n[0]:
> plotlist := [x, 0]:
> for j from 1 to iterations do
    y := evalf(F(x)):
    plotlist := plotlist, [x, y], [y, y]:
    x := y;
  od:
> P := plot([plotlist], colour = black, thickness = 2):
> C := plot({F(n), n}, n = 0..2.0):
> display({P, C});
```



Here we see that the solution rotates about the steady state as it approaches it.

Now increase a further.

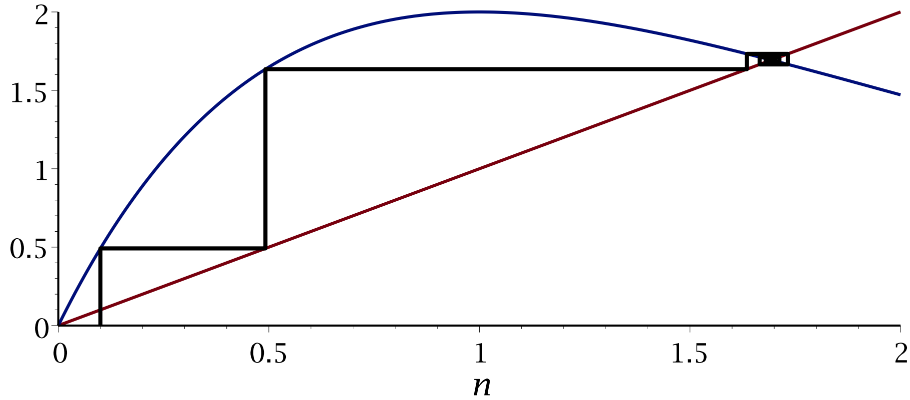
```
> a := 2·exp(1);
```

$$a := 2\,\mathrm{e} \tag{4}$$

```
> plotlist := NULL:
> x := n[0];
```

$$x := 0.1 \tag{5}$$

```
> plotlist := [x, 0]:
> for j from 1 to iterations do
    y := evalf(F(x)):
    plotlist := plotlist, [x, y], [y, y]:
    x := y;
  od:
> P := plot([plotlist], colour = black, thickness = 2):
> C := plot({F(n), n}, n = 0..2.0):
> display({P, C});
```



We see that the postive steady state is still stable, but the slope of the map at the steady state is negative and increasing in magnitude. So we expect that there is a value of a at which the steady state ceases to be stable.

d) We see from the Maple output that $h'(x^*)$ becomes increasingly larger in magnitude as a increases, and that $h'(x^*) < 0$ (where $x^* > 0$). So we look for the value of a at which

$$h'(x^*) = -1 \Leftrightarrow 1 - \ln(a) = -1$$

$$\Leftrightarrow \ln(a) = 2$$

$$\Leftrightarrow a = e^2$$

$\therefore$ for $1 < a < e^2$, the positive steady state at $x^* = \ln(a)$ is a stable steady state.

e) see Maple output

1(e). We check values of a larger than the critical value a = e^2.

> $a := 1.2 \cdot (\exp(1))^2;$

$$a := 1.2\,(\mathrm{e})^2 \qquad (6)$$

> $plotlist := NULL:$

> $x := n[0];$

$$x := 0.1 \qquad (7)$$

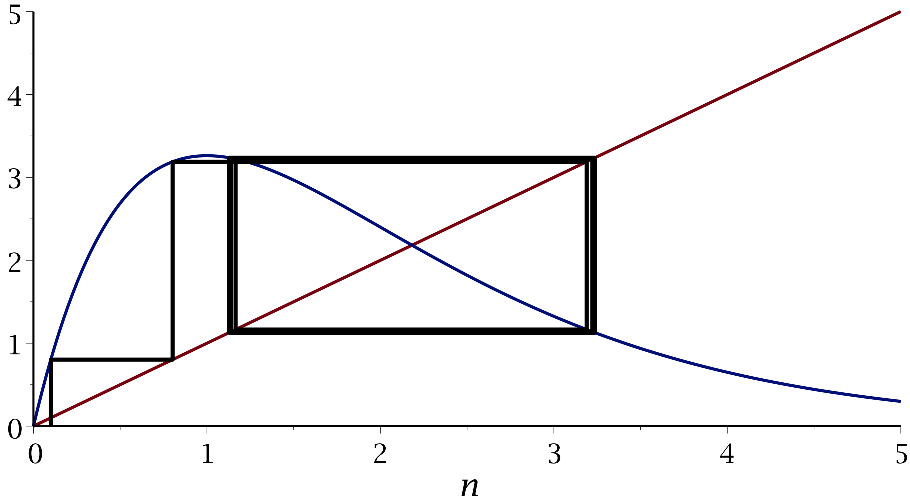
> $plotlist := [x, 0]:$

```
> for j from 1 to iterations do
    y := evalf(F(x)):
    plotlist := plotlist, [x, y], [y, y]:
    x := y;
  od:
```

> $P := plot([plotlist], colour = black, thickness = 2):$

> $C := plot(\{F(n), n\}, n = 0..5.0):$

> $display(\{P, C\});$

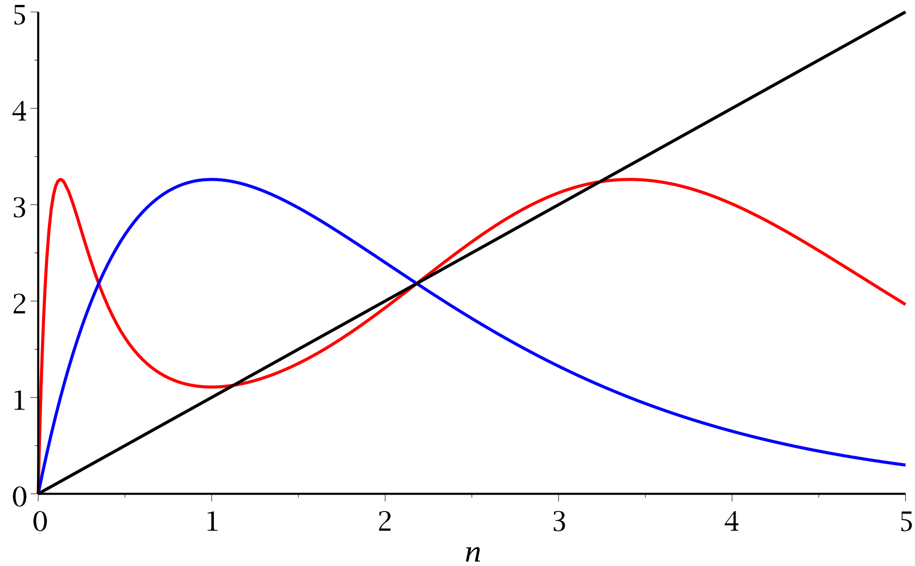


We obtain the required two-cycle.

> $F2 := n \rightarrow F(F(n));$

$$F2 := n \rightarrow F(F(n)) \qquad (8)$$

> $plot([F2(n), F(n), n], n = 0..5, colour = [red, blue, black]);$



The points where the red, blue, and black curves intersect correspond to the steady states of the first order map. The two additional points where the red and black curves intersect are the points that correspond to the two-cycle.

1(f) Exploring the behaviour of the map as a increases further.

> $a := 2\cdot(\exp(1))^2;$

$$a := 2\,(\mathrm{e})^2 \quad (9)$$

> $plotlist := NULL:$

> $x := n[0];$

$$x := 0.1 \quad (10)$$

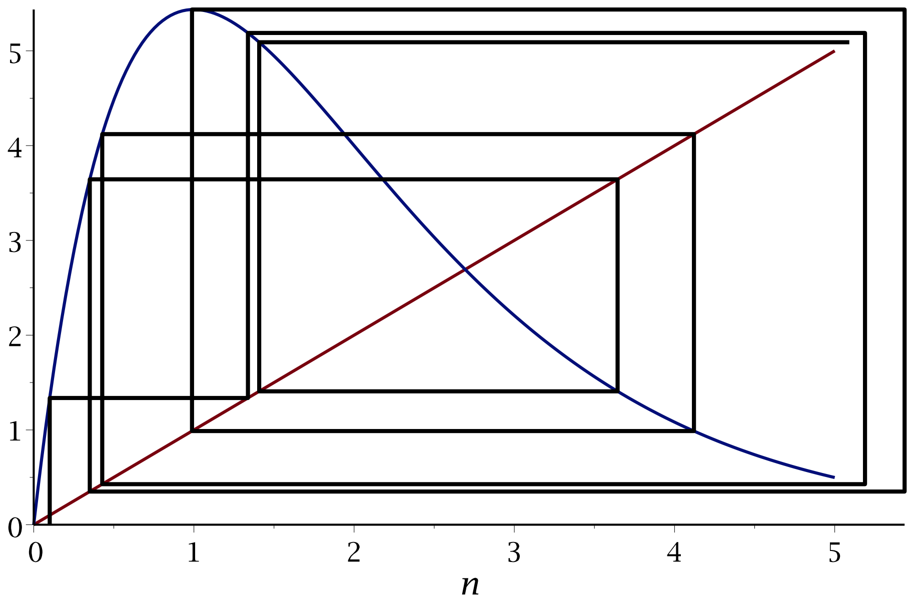
> $plotlist := [x, 0]:$

```
> for j from 1 to iterations do
   y := evalf(F(x)) :
   plotlist := plotlist, [x, y], [y, y] :
   x := y;
  od:
```

> $P := plot([plotlist], colour = black, thickness = 2):$

> $C := plot(\{F(n), n\}, n = 0..5.0):$

> $display(\{P, C\});$



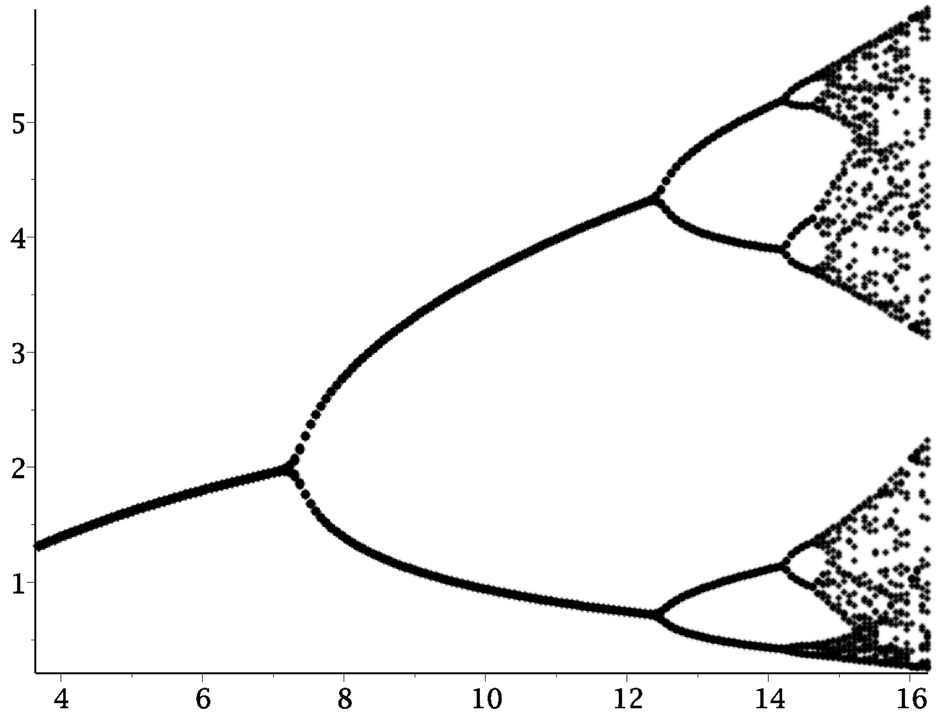
We tested a=1.2e^2, 1.8e^2, and 2e^2, and it looks as though chaos may have emerged by a=2e^2. We use this information to plot the bifurcation diagram.

> $plotlist := NULL: n[0] := 0.2; transiterations := 100;$

$$n_0 := 0.2$$

$$transiterations := 100 \quad (11)$$

```
> for q from 0.5 by 0.01 to 2.2 do
  a := q·exp(1)^2 :
  x := n[0] :
  for j from 1 to transiterations do
  y := evalf(F(x)) :
  x := y;
  od:
  for j from 1 to 50 do
  y := evalf(F(x)) :
  plotlist := plotlist, [a, x] :
  x := y;
  od:
  od:
> Q := pointplot([plotlist], colour = black, symbolsize = 1) :
> display(Q);
```



We see that the map undergoes a period-doubling cascade to chaos as a increases..

2. (Exercise 1.14)

$f(x) = x^2 + x$

fixed points:

$x^* = x^{*2} + x^* \iff x^* = x^*(x^*+1)$

$\iff x^* = 0 \text{ OR } x^* + 1 = 1 \iff x^* = 0$

Thus, there is only one fixed point + it is at $x^* = 0$.

Cobwebbing using Maple (see following page) we see that initial points $x_0 < 1$ and $x_0 > 0$ go to $+\infty$, while intial points $-1 \leq x_0 \leq 0$ go to the fixed point at 0.

(Thus the fixed point at 0 has mixed stability. The derivative at 0 is

$$f'(0) = (2x+1)\Big|_{x=0} = 1$$

Which is inconclusive + thus consistent with our mixed stability result.

```
> restart:
> with(plots):
> F := n -> n^2 + n;
```

$$F := n \to n^2 + n \quad (1)$$

2 - where non-fixed points go .

```
> n[0] := 0.1; iterations := 10;
```

$$n_0 := 0.1$$

$$iterations := 10 \quad (2)$$

```
> plotlist := NULL:
> x := n[0]:
> plotlist := [x, 0]:
> for j from 1 to iterations do
   y := evalf(F(x)):
   plotlist := plotlist, [x, y], [y, y]:
   x := y;
   od:
> P1 := plot([plotlist], colour = black, thickness = 2):
> n[0] := -0.5;
```

$$n_0 := -0.5 \quad (3)$$

```
> plotlist := NULL:
> x := n[0]:
> plotlist := [x, 0]:
> for j from 1 to iterations do
   y := evalf(F(x)):
   plotlist := plotlist, [x, y], [y, y]:
   x := y;
   od:
> P2 := plot([plotlist], colour = green, thickness = 2):
> n[0] := -0.8;
```

$$n_0 := -0.8 \quad (4)$$

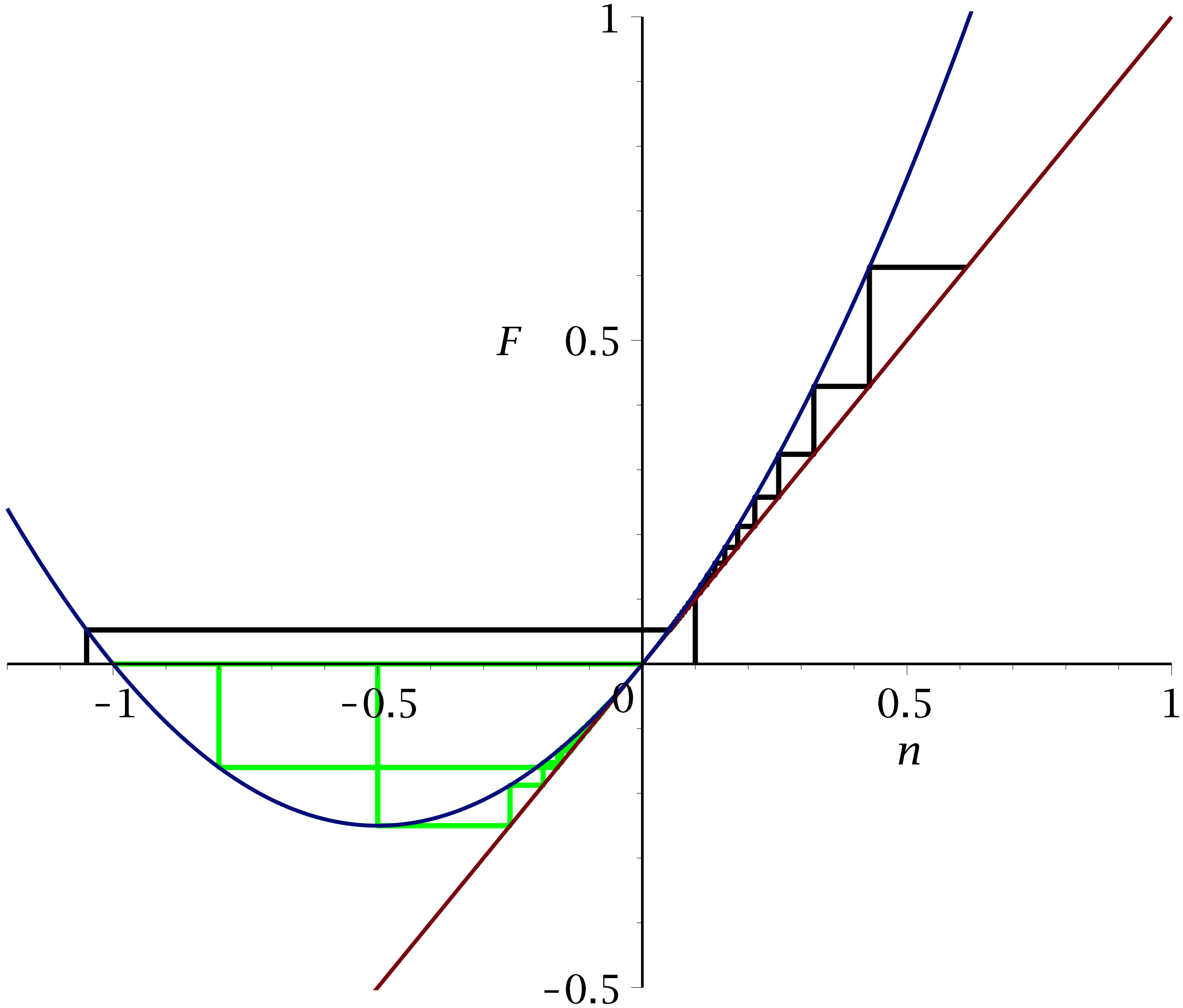
```
> plotlist := NULL:
> x := n[0]:
> plotlist := [x, 0]:
> for j from 1 to iterations do
   y := evalf(F(x)):
   plotlist := plotlist, [x, y], [y, y]:
   x := y;
   od:
> P3 := plot([plotlist], colour = green, thickness = 2):
> n[0] := -1;
```

$$n_0 := -1 \quad (5)$$

```
> plotlist := NULL:
> x := n[0]:
> plotlist := [x, 0]:
> for j from 1 to iterations do
   y := evalf(F(x)):
   plotlist := plotlist, [x, y], [y, y]:
   x := y;
  od:
> P4 := plot([plotlist], colour = green, thickness = 2):
> n[0] := -1.05;
```

$$n_0 := -1.05 \qquad (6)$$

```
> plotlist := NULL:
> x := n[0]:
> plotlist := [x, 0]:
> for j from 1 to iterations do
   y := evalf(F(x)):
   plotlist := plotlist, [x, y], [y, y]:
   x := y;
  od:
> P5 := plot([plotlist], colour = black, thickness = 2):
> C := plot({F(n), n}, n = -1.2..1.0, F = -0.5..1):
> display(P1, P2, P3, P4, P5, C);
```



1
F
0.5
-0.5
-1
0
n

>

3. (Exercise 2.5)

$f(x,y,z) = (x^2 y, y^2, xz + y)$

fixed points:

$$\begin{cases} x^* = x^{*2} y^* \\ y^* = y^{*2} \\ z^* = x^* z^* + y^* \end{cases} \iff \begin{cases} x^* = 0 \text{ or } x^* y^* = 1 \\ y^* = 0 \text{ or } y^* = 1 \\ z^* = \dfrac{y^*}{1 - x^*} \end{cases}$$

$\therefore$ the fixed points are $(0,0,0)$; $(0,1,1)$

stability:

Jacobian: $J = \begin{bmatrix} 2xy & x^2 & 0 \\ 0 & 2y & 0 \\ z & 1 & x \end{bmatrix}$

@ $(0,0,0)$: $J\Big|_{(0,0,0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

eigenvalues:

We can read these from the diagonal since the matrix is lower triangular.

We have

$\lambda_{1,2,3} = 0$.

So the fixed point is an attracting (stable) steady state.

@ $(0,1,1)$: $$J\Big|_{(0,1,1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

eigenvalues:

We can read these from the diagonal since the matrix is lower triangular. We have

$$\lambda_{1,2} = 0, \quad \lambda_3 = 2$$

$\therefore$ the fixed point is a <u>saddle</u>.