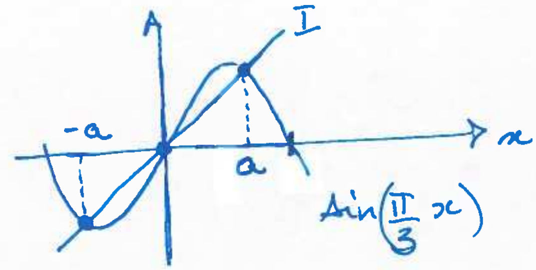
2017
Math 339 - Assignment #3
Solutions

1. Exercise 2.6

$$f(x,y) = \left(\sin\left(\tfrac{\pi}{3}x\right), \tfrac{y}{2}\right)$$

fixed pts:



$$\begin{cases} x^* = \sin\left(\frac{\pi}{3}x^*\right) \\ y^* = \frac{y^*}{2} \end{cases}$$

We see that there are three fixed points:

$(0,0)$; $(a,0)$; $(-a,0)$ where $a = \frac{1}{2}$ (found using Maple).

Stability:

$$Df = \begin{bmatrix} \frac{\pi}{3}\cos\left(\frac{\pi}{3}x\right) & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\therefore \lambda_1 = \frac{\pi}{3}\cos\left(\frac{\pi}{3}x\right), \quad \lambda_2 = \frac{1}{2}$$

So stability is determined by λ_1.

At $(0,0)$

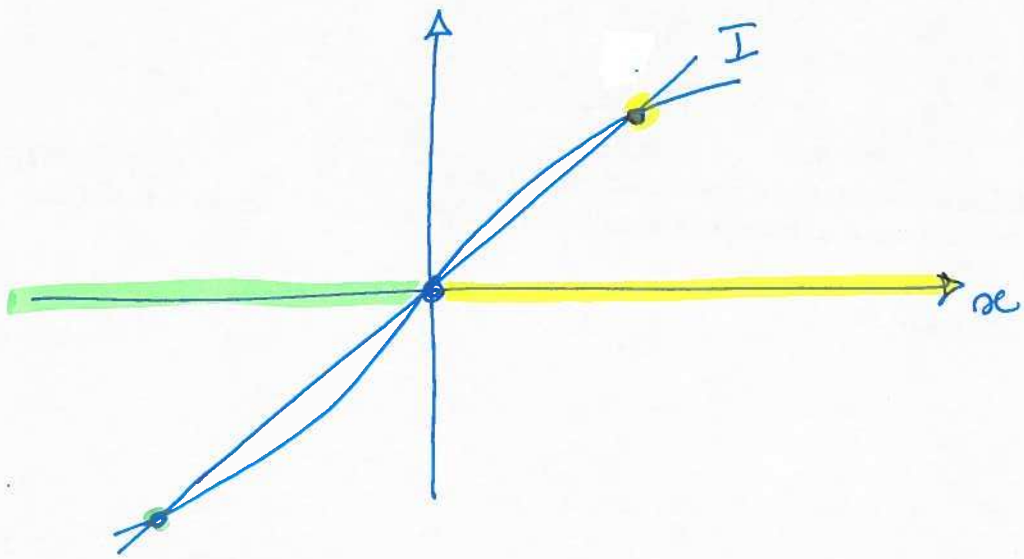
$\lambda_1 = \frac{\pi}{3}$ & so the steady state at the origin is unstable (saddle)

At $(\pm a, 0) = (\pm\frac{1}{2}, 0)$

$$\lambda_1 = \frac{\pi}{3}\cos\left(\frac{\pm\pi}{3}a\right) = \frac{\pi}{3}\cos\left(\frac{\pm\pi}{6}\right) = \frac{\pi}{3}\frac{\sqrt{3}}{2} = 0.91$$

$\therefore$ the other two steady states are stable.

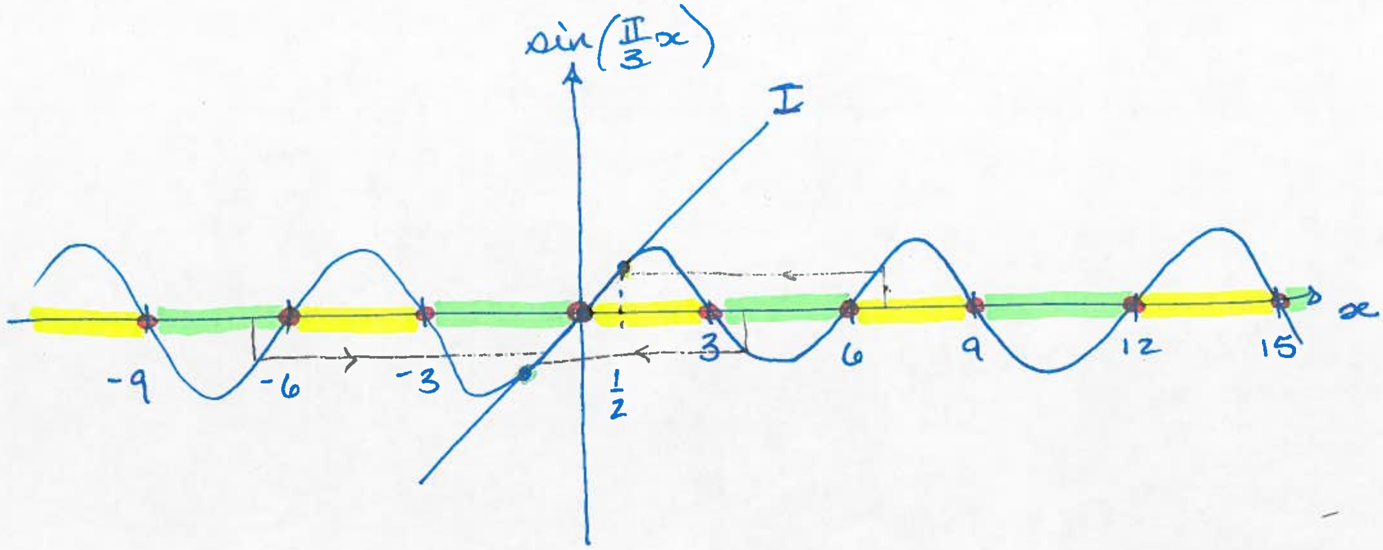
Basins of Attraction



- the basin of attraction for $(\frac{1}{2}, 0)$ is $\{(x,y) \mid x>0, y\in\mathbb{R}\}$
- " " " " " $(-\frac{1}{2}, 0)$ is $\{(x,y) \mid x<0, y\in\mathbb{R}\}$

(Maple work on next p)

What about the rest of the real line?



The basin of attraction of $(0,0)$ is $\{(x,y) \mid x = 3n, n \in \mathbb{Z}, y \in \mathbb{R}\}$

" " " " " $(\frac{1}{2},0)$ is $\{(x,y) \mid 6n < x < 6n+3, n \in \mathbb{Z}, y \in \mathbb{R}\}$

" " " " " $(-\frac{1}{2},0)$ is $\{(x,y) \mid 6n+3 < x < 6(n+1), n \in \mathbb{Z}, y \in \mathbb{R}\}$

> *restart*:

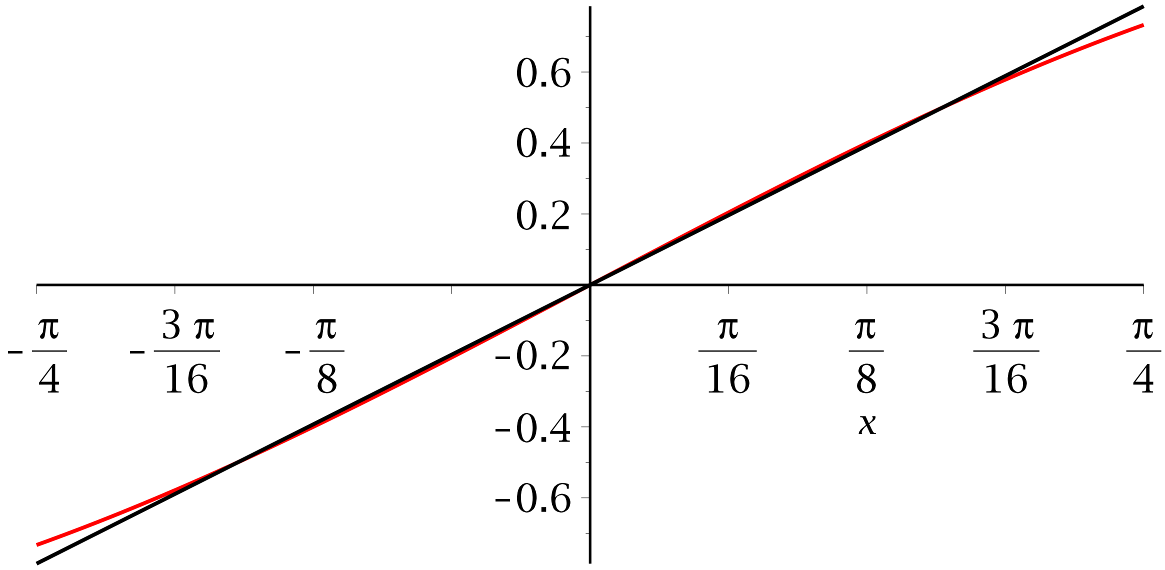
> *with*(*plots*) :

Calculations for 1. (Exercise 2.6)

> $F1 := x \to \sin\left(\frac{Pi}{3} \cdot x\right);$

$$F1 := x \to \sin\left(\frac{1}{3}\,\pi\,x\right) \qquad (1)$$

> $plot\left([F1(x), x], x = -\frac{Pi}{4} .. \frac{Pi}{4}, colour = [red, black]\right);$



> $fsolve(F1(x) - x = 0);$

$$0. \qquad (2)$$

> $fsolve\left(F1(x) - x = 0, x = \frac{Pi}{8} .. \frac{Pi}{4}\right);$

$$0.5000000000 \qquad (3)$$

> $evalf\left(\frac{Pi}{3} \cdot \frac{sqrt(3)}{2}\right);$

$$0.9068996827 \qquad (4)$$

>

2. Exercise 2.7

Hénon map w/ $b = 0.3$:

$$f(x, y) = (a - x^2 + 0.3y, x), \quad a > 0$$

a) fixed pts:

$$\begin{cases} x^* = a - x^{*2} + 0.3y^* & \dots \quad (1) \\ y^* = x^* & \dots \quad (2) \end{cases}$$

Combining (1) + (2) we obtain

$$x^* = a - x^{*2} + 0.3x^* \Leftrightarrow$$

$$\Leftrightarrow x^{*2} + 0.7x^* - a = 0$$

$$\Leftrightarrow x^* = \frac{-0.7 \pm \sqrt{0.49 + 4a}}{2} = p_{+,-}$$

The two fixed points are (p_+, p_+) ; (p_-, p_-).

Stability

$$Df\Big|_* = \begin{bmatrix} -2x^* & 0.3 \\ 1 & 0 \end{bmatrix}$$

eigenvalues:

$$(-2x^* - \lambda)(-\lambda) - 0.3 = 0 \iff (1)$$

$$(1) \iff \lambda^2 + 2x^*\lambda - 0.3 = 0$$

$$\iff \lambda = -x^* \pm \sqrt{x^{*2} + 0.3}$$

We see that for $x^* > 0$

$$\lambda_+ = -x^* + \sqrt{x^{*2} + 0.3} \simeq \sqrt{0.3} < 1$$

Thus, stability is determined by λ_-. For $x^* < 0$, stability is determined by λ_+.

Consider $x^* > 0$: $x^* = p_+$

$$\lambda_- = 0.35 - \sqrt{0.125 + a} - \sqrt{(-0.35 + \sqrt{0.125 + a})^2 + 0.3}$$

Consider $x^* < 0$: $x^* = p_-$

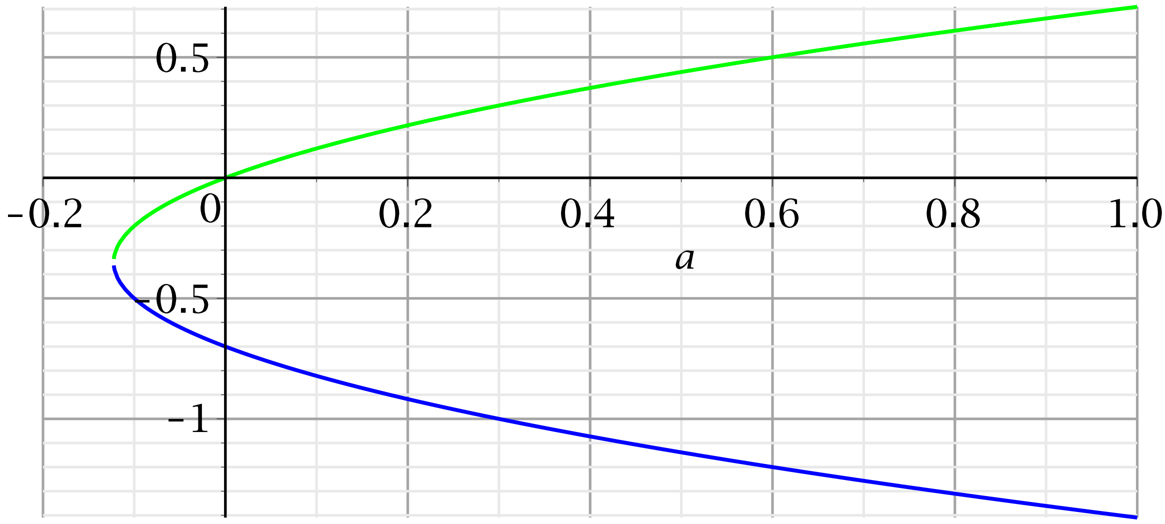
$$\lambda_+ = 0.35 + \sqrt{0.125 + a} + \sqrt{(-0.35 - \sqrt{0.125 + a})^2 + 0.3}$$

Calculations for 2 (Exercise 2.7)
(a) Finding the range of *a* values for which the map has one sink fixed point and one saddle fixed point.

> $xs\_m := -0.35 - \mathrm{sqrt}(0.1225 + a);\ xs\_p := -0.35 + \mathrm{sqrt}(0.1225 + a);$

$$xs\_m := -0.35 - \sqrt{0.1225 + a}$$
$$xs\_p := -0.35 + \sqrt{0.1225 + a} \tag{5}$$

> $plot([xs\_m(a), xs\_p(a)], a = -0.2..1, colour = [blue, green], gridlines);$



We see that x^*+ is positive (green curve), and x^*- is negative (blue curve) as expected.

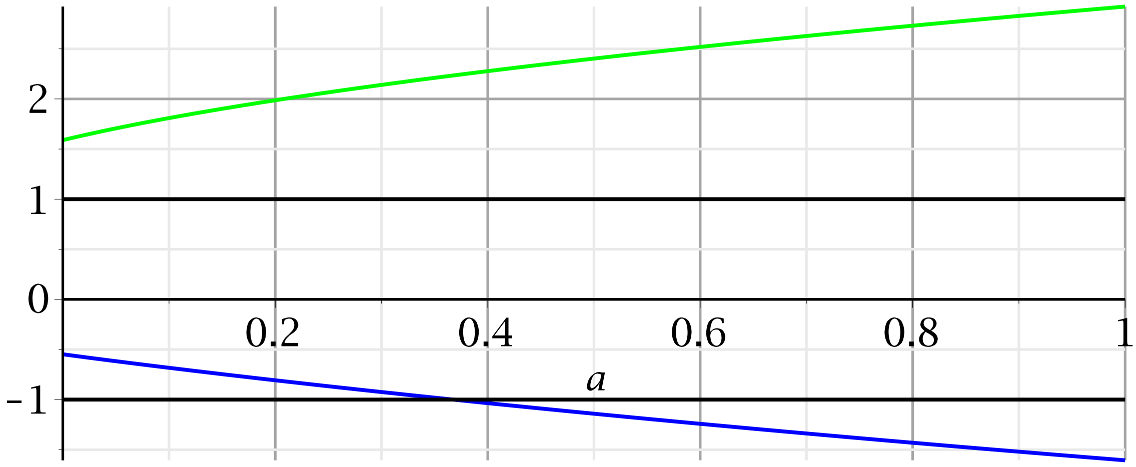
> $lambda\_m := a \to 0.35 - \mathrm{sqrt}(0.1225 + a) - \mathrm{sqrt}((-0.35 + \mathrm{sqrt}(0.1225 + a))^2 + 0.3);$

$$lambda\_m := a \to 0.35 - \sqrt{0.1225 + a} - \sqrt{\left(-0.35 + \sqrt{0.1225 + a}\right)^2 + 0.3} \tag{6}$$

> $lambda\_p := a \to 0.35 + \mathrm{sqrt}(0.1225 + a) + \mathrm{sqrt}((-0.35 - \mathrm{sqrt}(0.1225 + a))^2 + 0.3);$

$$lambda\_p := a \to 0.35 + \sqrt{0.1225 + a} + \sqrt{\left(-0.35 - \sqrt{0.1225 + a}\right)^2 + 0.3} \tag{7}$$

> $plot([lambda\_m(a), lambda\_p(a), 1, -1], a = 0..1, colour = [blue, green, black, black], gridlines);$



>

We see that *lambda_m* is initially less than 1 in magnitude, and becomes larger than 1 in magnitude for *a* a little less than 0.4. The other eigenvalue is always larger than 1 in magnitude. This result means that the negative steady state is a saddle for all $a>0$, and the positive steady state is a stable node for $0<a<c$, where c is a critical value close to 0.4.

We can find c numerically:

> $c := fsolve(lambda\_m(a) = -1);$

$$c := 0.3675000000 \tag{8}$$

Thus, the map has one sink fixed point and one saddle fixed point for $-0.1225<a<0.3675.$

>

$$H^2(x,y) = \left(a-(a-x^2+0.3y)^2+0.3x,\ a-x^2+0.3y\right)$$
$$= \left(a-a^2+2ax^2-0.6ay-x^4+0.6x^2y-0.09y^2+0.3x,\ a-x^2+0.3y\right)$$

Fixed pts:

$$\begin{cases} x^* = a-(a-x^{*2}+0.3y^*)^2+0.3x^* \\ y^* = a-x^{*2}+0.3y^* \end{cases} \Leftrightarrow \text{(III)}$$

$$\text{(III)} \Leftrightarrow \begin{cases} x^* = a-y^{*2}+0.3x^* \\ y^* = a-x^{*2}+0.3y^* \end{cases}$$

$$\Leftrightarrow \begin{cases} 0.7x^* = a-y^{*2} \\ 0.7y^* = a-x^{*2} \end{cases}$$

$$\Leftrightarrow \begin{cases} x^* = \frac{1}{0.7}\left(a-\frac{(a-x^{*2})^2}{0.49}\right) & \ldots\ldots (4) \\ y^* = \frac{a-x^{*2}}{0.7} & \ldots\ldots\ldots (5) \end{cases}$$

Working with (4) we obtain

$$0.7x^* = a-\frac{(a-x^{*2})^2}{0.49} \Leftrightarrow \text{(IV)}$$

$$\text{(IV)} \Leftrightarrow 0.343x^* = 0.49a-(a-x^{*2})^2$$

$$\Leftrightarrow 0.343x^* = 0.49a-\left[a^2-2ax^{*2}+x^{*4}\right] \Leftrightarrow \text{(V)}$$

$\therefore$ $\Leftrightarrow$ $0.343x^* = 0.49a - a^2 + 2ax^{*2} - x^{*4}$

$\Leftrightarrow$ $x^{*4} - 2ax^{*2} + 0.343x^* - 0.49a + a^2 = 0 \quad \cdots (6)$

We know that the fixed pts of H are also fixed pts of H^2, and so (6) should be factorable by (3):

$$
\begin{array}{r l}
 & x^{*2} - 0.7x^* + (0.49-a) \\
x^{*2} + 0.7x^* - a \,\big) & x^{*4} + 0x^{*3} - 2ax^{*2} + 0.343x^* - 0.49a + a^2 \\
-) & x^{*4} + 0.7x^{*3} - ax^{*2} \\
\hline
 & -0.7x^{*3} - ax^{*2} + 0.343x^* \\
-) & -0.7x^{*3} - 0.49x^{*2} + 0.7ax^* \\
\hline
 & (0.49-a)x^{*2} + (0.343-0.7a)x^* - 0.49a + a^2 \\
 & (0.49-a)x^{*2} + 0.7(0.49-a)x^* - a(0.49-a) \\
\hline
 & 0
\end{array}
$$

$\therefore$ (6) is factored as

$$\underbrace{(x^{*2} + 0.7x^* - a)}_{\text{The roots of this factor are the fixed points of } H.}\underbrace{(x^{*2} - 0.7x^* + 0.49 - a)}_{\text{The roots of this factor are the period-2 fixed points of } H.} = 0 \quad \cdots\cdots (7)$$

Solving the second factor of (7) we obtain

$$x^{*2} - 0.7x^* + (0.49 - a) = 0 \iff x^* = 0.35 \pm \sqrt{0.1225 + a - 0.49}$$

$$\iff x^*_{1,2} = 0.35 \pm \sqrt{a - 0.3675} \quad \ldots (8)$$

Thus, we must have $a \geq 0.3675$ for this two-cycle to exist, and $a = 0.3675$ is the critical value at which the sink fixed point of H (period 1 fixed point) becomes unstable (found in part (a))

The stability of the two-cycle is determined by the eigenvalues of

$$DH\Big|_{\vec{x}_1^*} DH\Big|_{\vec{x}_2^*} = \begin{bmatrix} -2x_1^* & 0.3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2x_2^* & 0.3 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4x_1^* x_2^* + 0.3 & -0.6x_1^* \\ -2x_2^* & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 2.26 - 4a & -0.210 - 0.6\sqrt{a - 0.3675} \\ -0.7 + 2\sqrt{a - 0.3675} & 0.3 \end{bmatrix}$$

(work shown on Maple)

(b) Finding the range of *a* for which the map has a period-two sink.

We expect the period-two sink to emerge when the sink fixed point becomes unstable, that is, at *a=0.365*. We begin by forming the second iterate map.

> $H := (x, y) \rightarrow (a - x^2 + 0.3 \cdot y, x);\ H2 := (x, y) \rightarrow H(H(x, y));$

$$H := (x, y) \rightarrow (a - x^2 + 0.3\, y, x)$$

$$H2 := (x, y) \rightarrow H(H(x, y)) \quad \textbf{(9)}$$

> $simplify(H2(x, y)[1]);$

$$a - a^2 + 2.\, a\, x^2 - 0.6\, a\, y - x^4 + 0.6\, x^2\, y - 0.09\, y^2 + 0.3\, x \quad \textbf{(10)}$$

Using the results above, we found the x-value of the period-2 fixed points to be

> $xs\_p2 := 0.35 + \mathrm{sqrt}(a - 0.3675);\ xs\_m2 := 0.35 - \mathrm{sqrt}(a - 0.3675);$

$$xs\_p2 := 0.35 + \sqrt{a - 0.3675}$$

$$xs\_m2 := 0.35 - \sqrt{a - 0.3675} \quad \textbf{(11)}$$

Stability of the two-cycle is determined by the Jacobian of the map H evaluated at each of the points in the two-cycle.

> $with(VectorCalculus) : with(LinearAlgebra) :$

> $J := Jacobian([H(x, y)[1], H(x, y)[2]], [x, y]);$

$$J := \begin{bmatrix} -2\, x & 0.3 \\ 1 & 0 \end{bmatrix} \quad \textbf{(12)}$$

> $A := simplify(expand(Multiply(subs(x = xs\_p2, J), subs(x = xs\_m2, J))));$

$$A := \begin{bmatrix} 2.2600 - 4.\, a & -0.210 - 0.6\sqrt{a - 0.3675} \\ -0.70 + 2\sqrt{a - 0.3675} & 0.3 \end{bmatrix} \quad \textbf{(13)}$$

> $EA := Eigenvalues(A);$

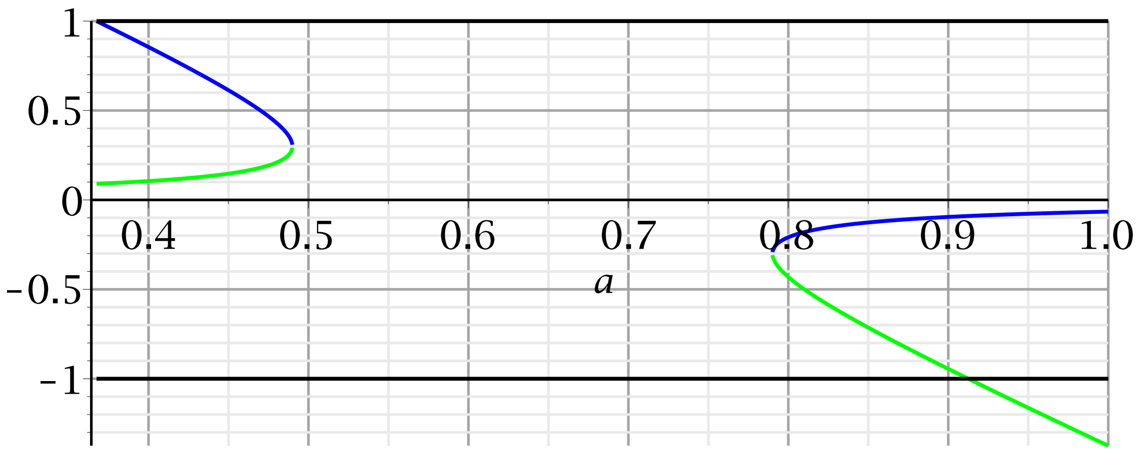
$$EA := \begin{bmatrix} -2.\, a + 1.280000000 + 0.02000000000\sqrt{10000.\, a^2 - 12800.\, a + 3871.} \\ -2.\, a + 1.280000000 - 0.02000000000\sqrt{10000.\, a^2 - 12800.\, a + 3871.} \end{bmatrix} \quad \textbf{(14)}$$

> $EA1 := unapply(EA[1], a);\ EA2 := unapply(EA[2], a);$

$$EA1 := a \rightarrow -2.\, a + 1.280000000 + 0.02000000000\sqrt{10000.\, a^2 - 12800.\, a + 3871.}$$

$$EA2 := a \rightarrow -2.\, a + 1.280000000 - 0.02000000000\sqrt{10000.\, a^2 - 12800.\, a + 3871.} \quad \textbf{(15)}$$

> $plot([EA1(a), EA2(a), 1, -1], a = 0.3675..1, colour = [blue, green, black, black], gridlines);$



We see that only the second eigenvalue (the one in green) becomes larger than 1 in magnitude by crossing the -1 line. Solving for this intersection point we find

> $fsolve(EA2(a) = -1);$

$$0.9125000000 \quad (16)$$

Thus, the two-cycle becomes unstable at *a=0.9125,* that is, the map has a period-2 sink for *0.3675<a<0.9125.*

3. (T2.9) with

$$f(x,y) = \left(\frac{x}{3},\ y - \frac{2}{3}x^2\right)$$

a) fixed pts

$$\begin{cases} x^* = \dfrac{x^*}{3} \\ y^* = y^* - \dfrac{2}{3}x^{*2} \end{cases} \Leftrightarrow \begin{cases} x^* = 0 \\ y^* = y^* \end{cases}$$

$\therefore$ Every point on the y axis is a fixed pt.

Find f^{-1}:

$$\begin{cases} x_{n+1} = \dfrac{x_n}{3} \\ y_{n+1} = y_n - \dfrac{2}{3}x_n^2 \end{cases} \Leftrightarrow \begin{cases} x_n = 3x_{n+1} \\ y_n = y_{n+1} + \dfrac{2}{3}x_n^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_n = 3x_{n+1} \\ y_n = y_{n+1} + \dfrac{2}{3}(3x_{n+1})^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_n = 3x_{n+1} \\ y_n = y_{n+1} + 6x_{n+1}^2 \end{cases}$$

$$\therefore f^{-1}(x,y) = (3x,\ y+6x^2)$$

b) Show that

$$S = \{(x, rx^2) : x \in \mathbb{R}\}$$

is invariant, for some r, for the map f.

$$f(x, rx^2) = \left(\frac{x}{3},\ rx^2 - \frac{2}{3}x^2\right) = \left(\frac{x}{3},\ \left(r-\frac{2}{3}\right)x^2\right)$$

For the set to be invariant, we require

$$\left(r-\frac{2}{3}\right)x^2 = r\left(\frac{x}{3}\right)^2 \Leftrightarrow rx^2 - \frac{2}{3}x^2 = r\frac{x^2}{9} \Leftrightarrow \sqrt{\,}$$

$$\Leftrightarrow r - \frac{2}{3} = \frac{r}{9} \Leftrightarrow 9r - 6 = r \Leftrightarrow 8r - 6 = 0$$

$$\Leftrightarrow r = \frac{6}{8} = \frac{3}{4}$$

Now check for invariance in f^{-1}:

$$f^{-1}\left(x, \frac{3}{4}x^2\right) = \left(3x,\ \frac{3}{4}x^2 + 6x^2\right) = \left(3x,\ \left(\frac{3}{4}+6\right)x^2\right)$$

$$= \left(3x,\ \left(\frac{3+24}{4}\right)x^2\right) = \left(3x,\ \frac{27}{4}x^2\right)$$

$$= \left(3x,\ \frac{3}{4}(9x^2)\right) = \left(3x,\ \frac{3}{4}(3x)^2\right) \in S$$

$\therefore$ S is invariant wrt f & f^{-1}.

c) Observe that given a starting point (x, rx^2),

$$f(x, rx^2) = \left(\frac{x}{3}, \frac{1}{12}x^2\right)$$

and so, $\because$ the x-coordinate has been reduced by 1/3 and we are still on S, then iteration of the map moves the starting point closer to $(0,0)$.

d) (see Maple work, next p)

3. Exercise T2.9 with the map *f(x,y) = (x/3, y-2/3 x^2).*

> $restart : with(plots) :$

> $F := (m, n) \rightarrow \left(\frac{m}{3}, n - \frac{2 \cdot m^2}{3} \right);$

$$F := (m, n) \rightarrow \left(\frac{1}{3}\, m,\ n - \frac{2}{3}\, m^2 \right) \quad \textbf{(17)}$$

> $iterations := 10;$

$$iterations := 10 \quad \textbf{(18)}$$

> $solnum := 0; plotlist := NULL;$

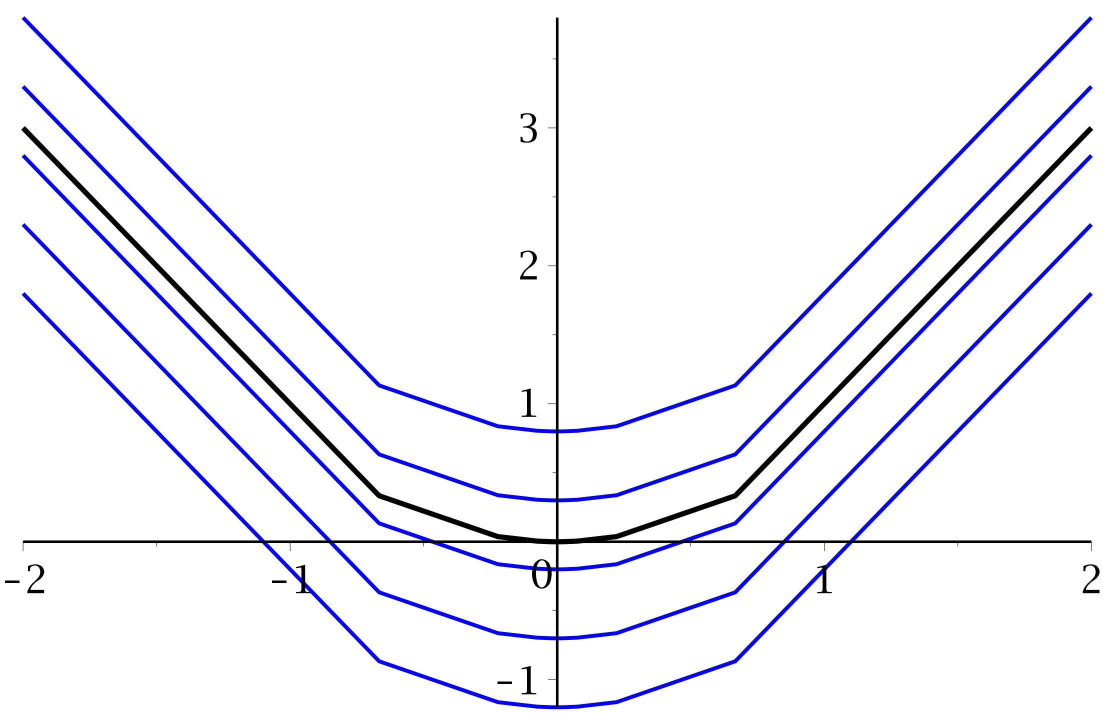
$$solnum := 0 \quad \textbf{(19)}$$

```
> for n0 from 1.8 by 0.5 to 3.8 do
   for m0 from -2 by 4 to 2 do
   orbit := NULL:
   solnum := solnum + 1;
   x := m0: y := n0:
   orbit := orbit, [x, y]:
  for j from 1 to iterations do
   (x1, y1) := evalf(F(x, y)):
   orbit := orbit, [x1, y1]:
   x := x1; y := y1;
   od:
  P[solnum] := plot([orbit], colour = blue, thickness = 1):
   od: od:
```

Now generate lists of points for the stable manifold

```
> for m0 from -2 by 4 to 2 do
   orbit := NULL:
   solnum := solnum + 1;
   x := m0: y := 3·m0^2/4 :
   orbit := orbit, [x, y];
   for j from 1 to iterations do
  (x1, y1) := evalf(F(x, y)):
   orbit := orbit, [x1, y1]:
   x := x1; y := y1;
   od:
  P[solnum] := plot([orbit], colour = black, thickness = 2):
   od:
```

> $plots:-display(entries(P,'nolist'));$



The plot above suggests that all trajectories are parallel to the set S, and terminate at the vertical axis, but only those points starting on the set S (in black) end up at the origin.

>

>

Consider any starting point (x,y) where $y - \frac{3}{4}x^2 = \varepsilon$. Then our Maple work suggests that the set

$$S_\varepsilon \{(x, rx^2+\varepsilon) : x \in \mathbb{R}\}$$

is invariant for the map f. Let's check this:

$$f(x,y) = \left(\frac{x}{3}, y - \frac{2}{3}x^2\right) = \left(\frac{x}{3}, \varepsilon + \frac{3}{4}x^2 - \frac{2}{3}x^2\right)$$

$$= \left(\frac{x}{3}, \frac{x^2}{12} + \varepsilon\right) = \left(\frac{x}{3}, \left(\frac{x}{3}\right)^2 \frac{3}{4} + \varepsilon\right) \in S_\varepsilon$$

$\therefore$ the only initial points (x_0, y_0) on orbits tending to $(0,0)$ are on S. All other orbits tend to points on the y axis, $(0,\varepsilon)$, where $\varepsilon = y_0 - \frac{3}{4}x_0^2$.

There is no unstable manifold.