Math 339 - 2017
A#4 - Sol'ns

1) 7.2

$$x'' + 3x' - 4x = 0 \quad \ldots \quad (1)$$

a) Let $y = x'$, then (1) is equivalent to

$$\begin{cases} x' = y \\ y' = -3y + 4x \end{cases}$$

b) This is a linear system, & so there is only one steady state, & it is at $(x,y) = (0,0)$.

The eigenvalues at $(0,0)$ are given by

$$|J - \lambda I| = 0 \iff \begin{vmatrix} -\lambda & 1 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$\iff \lambda(3+\lambda) - 4 = 0$$

$$\iff \lambda^2 + 3\lambda - 4 = 0$$

$$\iff \lambda = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2}$$

$$= 1 \text{ or } -4$$

Thus $(0,0)$ is a saddle. The eigenvectors are

$\underline{\lambda = 1}$

$$J\vec{u} = \lambda\vec{u} \iff \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\iff \begin{cases} u_2 = u_1 \\ 4u_1 - 3u_2 = u_2 \end{cases} \iff \begin{cases} u_2 = u_1 \\ u_1 = u_2 \end{cases}$$ Redundant system (as expected)

choose $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

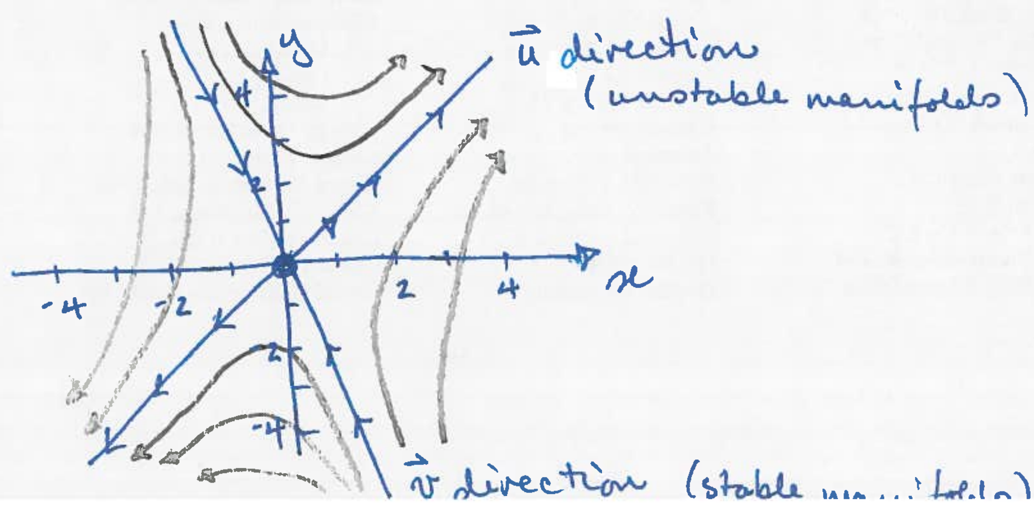
$\underline{\lambda = -4}$

$$J\vec{v} = \lambda\vec{v} \iff \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -4\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\iff \begin{cases} v_2 = -4v_1 \\ 4v_1 - 3v_2 = -4v_2 \end{cases} \iff \begin{cases} v_2 = -4v_1 \\ 4v_1 = -v_2 \end{cases}$$ Redundant system (as expected).

choose $\vec{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

Sketch:



2) 7.3

$$\begin{cases} \dot{x} = 2x - y \\ \dot{y} = x^2 + 4y \end{cases}$$

This is a nonlinear system. To find the equilibria, we solve

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - y = 0 \\ x^2 + 4y = 0 \end{cases} \Leftrightarrow \begin{cases} y = 2x \\ x^2 + 8x = 0 \end{cases} \Leftrightarrow \begin{cases} y = 2x \\ x(x+8) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \text{ or } \begin{cases} y = -16 \\ x = -8 \end{cases}$$

So the steady states are $(0,0)$ & $(-8,-16)$.

Stability is given by the eigenvalues:

$\underline{(0,0)}$

$$\left| J - \lambda I \right|_{(0,0)} = 0 \Leftrightarrow \begin{vmatrix} 2-\lambda & -1 \\ 2\cdot 0 & +4-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)(4-\lambda) - 0 = 0$$

$$\Leftrightarrow \lambda_1 = 2 \,,\ \lambda_2 = 4$$

$\therefore$ $(0,0)$ is unstable.

$(-8,-16)$

$$|J-\lambda I| = 0 \Leftrightarrow \begin{vmatrix} 2-\lambda & -1 \\ 2\cdot(-8) & 4-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (2-\lambda)(4-\lambda) - 16 = 0$$

$$\Leftrightarrow \lambda^2 - 6\lambda + 8 - 16 = 0$$

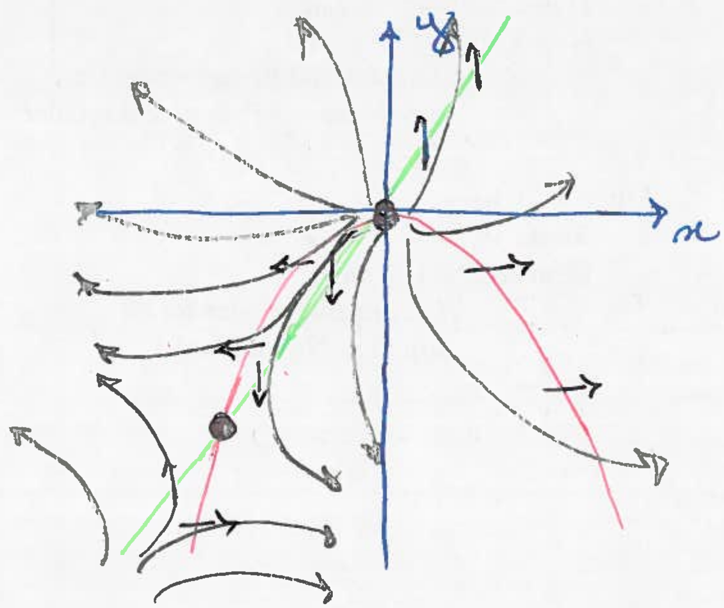
$$\Leftrightarrow \lambda^2 - 6\lambda - 8 = 0$$

$$\Leftrightarrow \lambda = 3 \pm \sqrt{9+8} = 3 \pm \sqrt{17}$$

$\because \sqrt{17} > 3$, we see that $\lambda_1 > 0$ + $\lambda_2 < 0$.

$\therefore (-8,-16)$ is a saddle node + is therefore unstable.

The phase plane sketch was not asked for, but I include it here:



$\dot{x} = 0 \Leftrightarrow y = 2x$

$\dot{y} = 0 \Leftrightarrow y = -\frac{1}{4}x^2$

On $\dot{x}=0$, $y=2x$, so

$$\dot{y} = x^2 + 4y$$
$$= x^2 + 4(2x)$$
$$= x^2 + 8x$$
$$= x(x+8)$$

$\therefore \dot{y} > 0$ for $x > 0$ + $x < -8$

+ $\dot{y} < 0$ for $-8 < x < 0$

On $\dot{y}=0$, $y=-\frac{1}{4}x^2$, so

$$\dot{x} = 2x - y = 2x + \frac{1}{4}x^2 = x\left(2+\frac{x}{4}\right) = \frac{x}{4}(8+x)$$

$\therefore$ $\dot{x} > 0$ for $x>0$ or $x<-8$

$\dot{x} < 0$ for $-8<x<0$

3) 7.5b

$$\begin{cases} \dot{x} = -2x + 3y \\ \dot{y} = 7x - 6y \end{cases}$$

This is a linear system, & so it has one steady state that is at $(0,0)$. The stability of this equilibrium is given by the eigenvalues:

$$|J-\lambda I| = 0 \Leftrightarrow \begin{vmatrix} -2-\lambda & 3 \\ 7 & -6-\lambda \end{vmatrix} = 0 \Leftrightarrow (-2-\lambda)(-6-\lambda) - 21 = 0$$

$$\Leftrightarrow \lambda^2 + 8\lambda + 12 - 21 = 0 \Leftrightarrow \lambda^2 + 8\lambda - 9 = 0$$

$$\Leftrightarrow \lambda = -4 \pm \sqrt{16+9} = -4 \pm \sqrt{25} = 1, -9$$

$\therefore$ the steady state is a saddle.

The eigenvectors are given by:

$\lambda_1 = 1$

$$J\vec{u} = \lambda\vec{u} \Leftrightarrow \begin{bmatrix} -2 & 3 \\ 7 & -6 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 1\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Leftrightarrow$$

$$\lambda_1 \iff \begin{cases} -2u_1 + 3u_2 = u_1 \\ 7u_1 - 6u_2 = u_2 \end{cases} \iff \begin{cases} u_1 = u_2 \\ u_1 = u_2 \end{cases}$$ Redundant system, as required.

choose $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

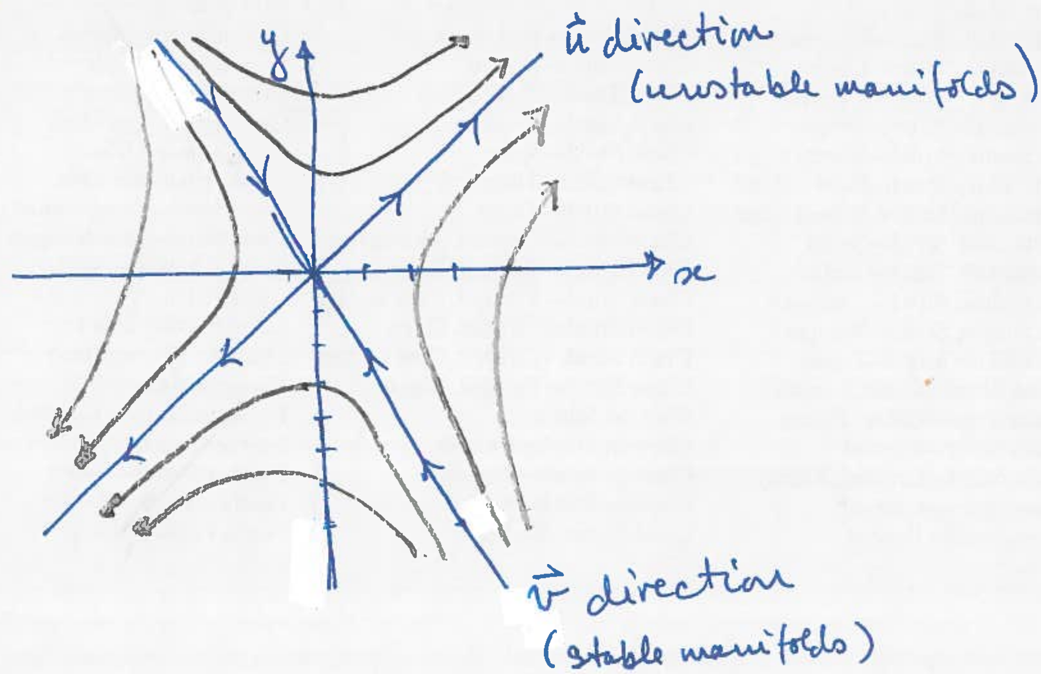
$\underline{\lambda_2 = -9}$

$$J\vec{v} = \lambda\vec{v} \iff \begin{bmatrix} -2 & 3 \\ 7 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -9\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\iff \begin{cases} -2v_1 + 3v_2 = -9v_1 \\ 7v_1 - 6v_2 = -9v_2 \end{cases} \iff \begin{cases} 7v_1 = -3v_2 \\ 7v_1 = -3v_2 \end{cases}$$ Redundant, as required

Choose $\vec{v} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

<u>Sketch</u>



4) 7.8(b)

$$\ddot{x} + x + x^3 = 0$$

Let $y = \dot{x}$, then we have

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - x^3 \end{cases}$$

Steady states & nullclines are given by:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ -x(1+x^2) = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$$

$\therefore$ (0,0) is the only steady state.

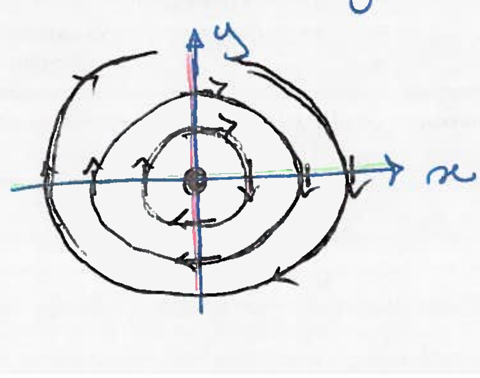
Stability:

$$|J - \lambda I| = 0 \Leftrightarrow \left| \begin{vmatrix} 0-\lambda & 1 \\ -1-3x^2 & 0-\lambda \end{vmatrix} \right|_{x=0} = 0$$

$$\Leftrightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda^2 + 1 = 0 \Leftrightarrow \lambda = \pm i$$

$\therefore$ (0,0) is a centre, according to the linear analysis.



On $\dot{x} = 0$, $y = 0$

$\dot{y} = -x(1+x^2)$

$\therefore \dot{y} < 0$ for $x > 0$

> 0 for $x < 0$

On $\dot{y} = 0$, $x = 0$

$\dot{x} = y$ > 0 for $y >$ [illegible]

< 0 for $y <$ [illegible]