Math 339 - 2017
Assignment #5 - Sol'ns

1. a) Require

i) $\mathcal{L}\left(\frac{d}{\alpha b}, \frac{a}{b}\right) = 0 \iff a \ln\left(\frac{a}{b}\right) - b\frac{a}{b} - \alpha b \frac{d}{\alpha b} + d\ln\left(\frac{d}{\alpha b}\right) + K' = 0$

$$\iff a\left(\ln\left(\tfrac{a}{b}\right) - 1\right) + d\left(\ln\left(\tfrac{d}{\alpha b}\right) - 1\right) + K' = 0$$

$$\iff K' = a\left(1 - \ln\left(\tfrac{a}{b}\right)\right) + d\left(1 - \ln\left(\tfrac{d}{\alpha b}\right)\right)$$

iii) $\dot{\mathcal{L}}(x, y) \le 0 \iff \frac{a}{y}\dot{y} - b\dot{y} - \alpha b\dot{x} + \frac{d}{x}\dot{x} \le 0$

$$\iff \left(\frac{a}{y} - b\right)(\alpha bxy - dy) + \left(\frac{d}{x} - \alpha b\right)(ax - bxy)$$

$$\iff (a - by)(\alpha bx - d) + (d - \alpha bx)(a - by) \le 0$$

$$\iff 0 \le 0 \quad \checkmark$$

ii) We know that the curves $\mathcal{L}(x,y) = C$ are concentric periodic orbits, and $\dot{\mathcal{L}} = 0$ everywhere in the 1st quadrant. So we only need to show that $\mathcal{L}(x,y)$ is strictly positive or strictly negative in the first quadrant, with the exception of the equilibrium point where $\mathcal{L}(x,y) = 0$.

Let $x = \frac{d}{\alpha b} \cdot r$, $y = \frac{a}{b} \cdot r$. Then

$$\mathcal{L}(x,y) = a \ln\left(r \frac{a}{b}\right) - b r \frac{a}{b} - \alpha b \frac{d}{\alpha b} r + d \ln\left(\frac{d}{\alpha b} r\right) + k'$$

$$= a \ln(r) + a \ln\left(\frac{a}{b}\right) - ar - dr + d \ln(r) + d \ln\left(\frac{d}{\alpha b}\right) + k'$$

$$= a \ln(r) + d \ln(r) + a\left(\cancel{\ln\left(\frac{a}{b}\right)} - r\right) + d\left(\cancel{\ln\left(\frac{d}{\alpha b}\right)} - r\right) + a\left(1 - \cancel{\ln\left(\frac{a}{b}\right)}\right) + d\left(1 - \cancel{\ln\left(\frac{d}{\alpha b}\right)}\right)$$

$$= a \ln(r) - ar + d \ln(r) - dr + a + d$$

$$= a(\ln(r) - r + 1) + d(\ln(r) - r + 1)$$

$$= (a+d)(\ln(r) - r + 1) < 0 \ \forall r > 0, r \neq 1$$

$\therefore \mathcal{L}(x,y) < 0 \ \forall \ x>0 \ \& \ y>0$, except at the equilibrium point.

$\therefore$ $\mathcal{L}(x,y)$ is a Lyapunov function for the coexistence steady state of (1) if

$$K' = a\left(1-\ln\left(\frac{a}{b}\right)\right) + d\left(1-\ln\left(\frac{d}{\alpha b}\right)\right).$$

b) i) $\mathcal{L}_1\left(\frac{d}{\alpha b}, \frac{a}{b}\right) = 0$, so $\mathcal{L}_1$ satisfies this requirement

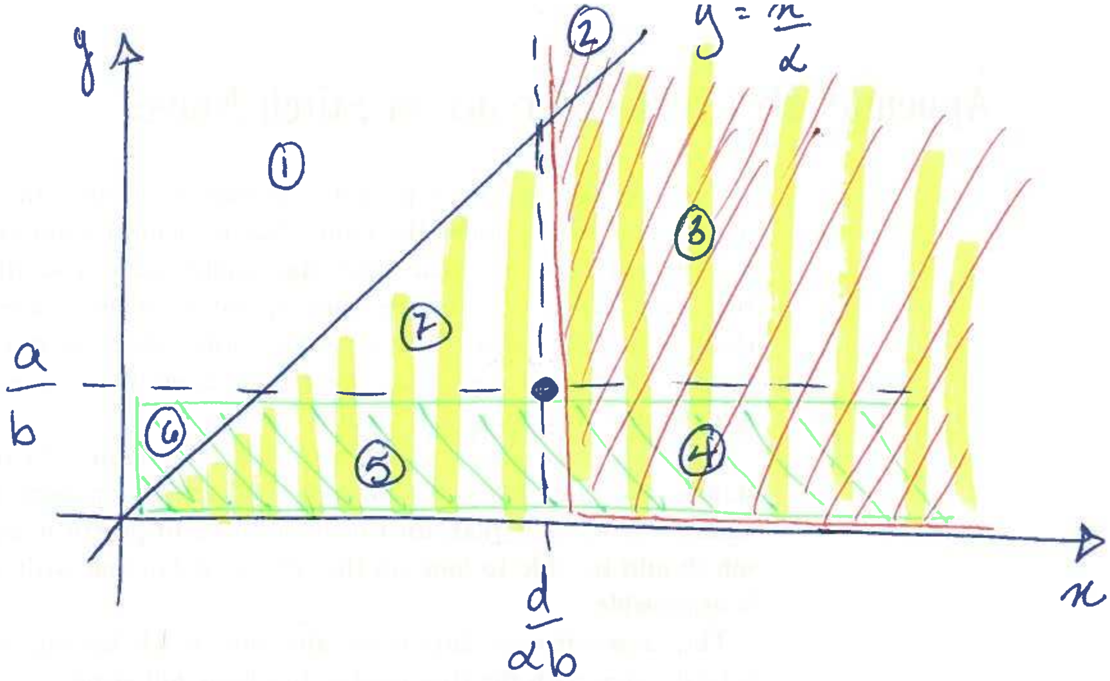
ii) $\mathcal{L}_1(x,y) > 0 \ \forall \ (x,y) \neq \left(\frac{b}{\alpha d}, \frac{a}{b}\right)$, so $\mathcal{L}_1$ satisfies this requirement

iii) $\dot{\mathcal{L}}_1(x,y) = 2\left(x - \frac{d}{\alpha b}\right)(ax - bxy) + 2\left(y - \frac{a}{b}\right)(\alpha bxy - dy)$

$$= \frac{2}{\alpha b}(\alpha bx - d)(ax - bxy) + \frac{2}{b}(by - a)(\alpha bxy - dy)$$

$$= \frac{2x}{\alpha b}(\alpha bx - d)(a - by) + \frac{2y}{b}(by - a)(\alpha bx - d)$$

$$= \frac{2}{b}\left(\frac{x}{\alpha} - y\right)\left[(\alpha bx - d)(a - by)\right]$$



= $a - by > 0$

= $\alpha b x - d > 0$

= $\frac{x}{\alpha} - y > 0$

$\therefore$ in regions 1, 3, 5, and 6, $\mathcal{L}_1(x,y) < 0$. In the remaining regions, $\mathcal{L}_1(x,y) > 0$, and so $\mathcal{L}_1(x,y)$ is NOT a Lyapunov function for the coexistence steady state of (1).

2. a) The phase plane equation is

$$\frac{\dot{y}}{\dot{x}} = \left(\frac{ay^2}{x} - y\right) \frac{1}{ayx - x^2} \quad \Leftrightarrow \quad 1/_{1}$$

$$1/_{1} \Leftrightarrow \quad \frac{dy}{dx} = \frac{y}{x}\,(ay - x)\,\frac{1}{x(ay - x)}$$

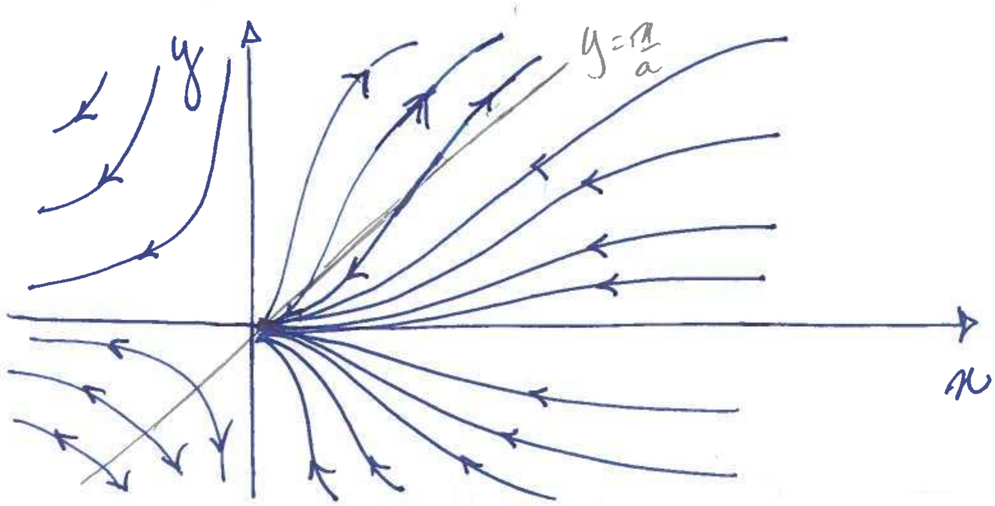
$$\Leftrightarrow \quad \frac{dy}{dx} = \frac{y}{x^2}, \quad ay - x \neq 0$$

$$\Leftrightarrow \quad \int \frac{1}{y}\,dy = \int \frac{1}{x^2}\,dx, \quad ay - x \neq 0$$

$$\Leftrightarrow \quad \ln|y| = -\frac{1}{x} + \tilde{K}, \quad x \neq ay$$

$$\Leftrightarrow \quad |y| = K e^{-\frac{1}{x}}, \quad x \neq ay$$

$$\Leftrightarrow \quad y = \pm K e^{-\frac{1}{x}}, \quad x \neq ay$$

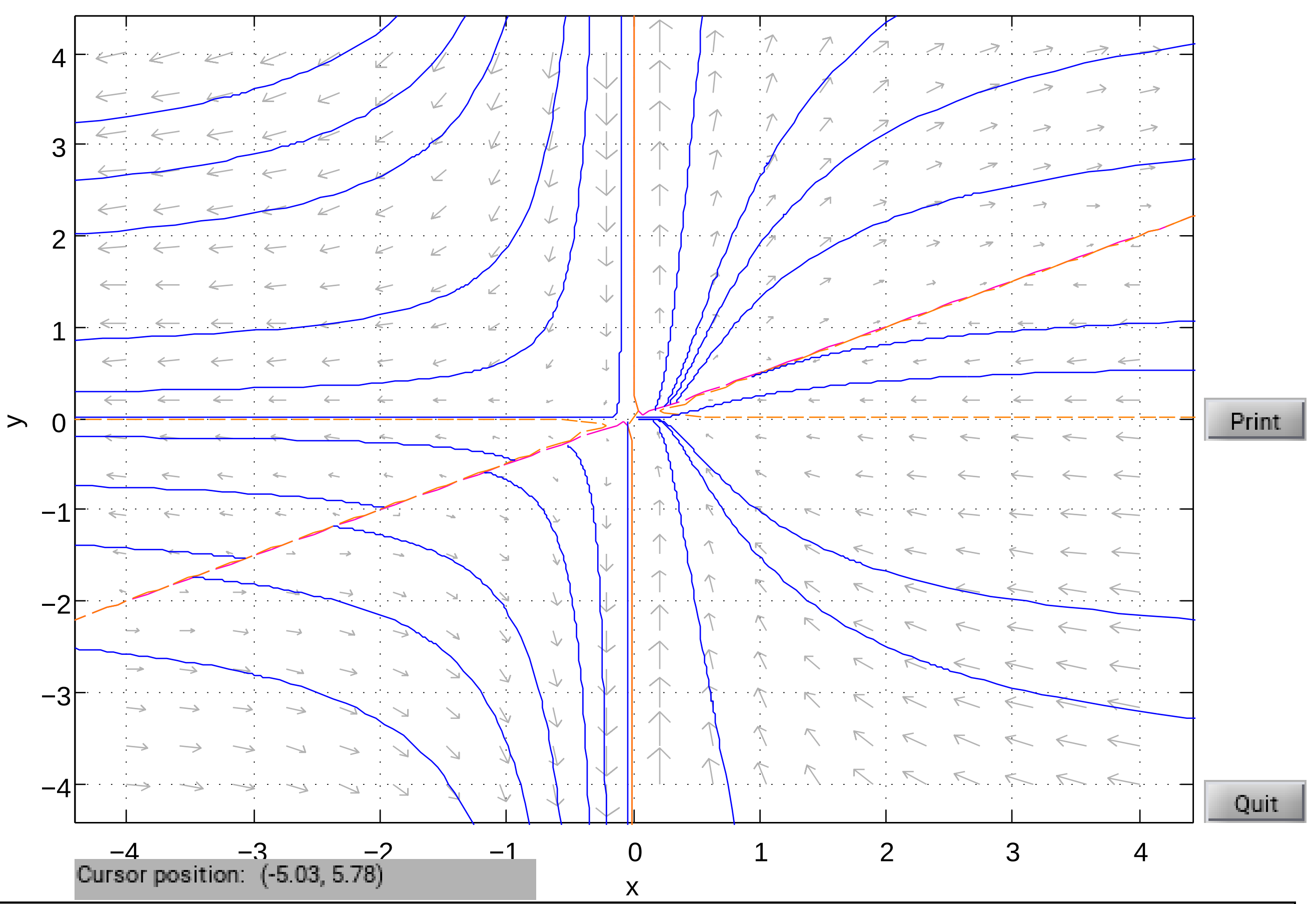


$, x_0 \neq 0$

(see phase plane next p.)

$x' = x\,(a\,y - x)$
$y' = y\,(a\,y - x)/x$

$a = 2$



The backward orbit from (-3.1, 3.5) left the computation window.
Ready.
The forward orbit from (-4.1, -1.4) left the computation window.
The backward orbit from (-4.1, -1.4) --> a possible eq. pt. near (-3.1, -1.5).
Ready.

Note: flow direction along these curves is determined by looking at $\dot{x}$ & $\dot{y}$.

3. $\begin{cases} \dot{x} = x - 3y - 4 \\ \dot{y} = x^2 + 4y \end{cases}$

a) Steady states:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Leftrightarrow \begin{cases} x - 3y - 4 = 0 \\ x^2 + 4y = 0 \end{cases} \Leftrightarrow \begin{cases} y = \frac{1}{3}x - \frac{4}{3} \\ y = -\frac{1}{4}x^2 \end{cases}$$

Combining, we obtain:

$$\frac{1}{3}x - \frac{4}{3} = -\frac{1}{4}x^2 \Leftrightarrow 4x - 16 = -3x^2 \Leftrightarrow$$

$$\Leftrightarrow 3x^2 + 4x - 16 = 0 \Leftrightarrow x = \frac{-2 \pm \sqrt{4 + 48}}{3}$$

$$\Leftrightarrow x = \frac{-2 \pm \sqrt{52}}{3} = \frac{-2 \pm 2\sqrt{13}}{3} = \frac{2}{3}\left(-1 \pm \sqrt{13}\right)$$

$$\text{and } y = \frac{2}{9}\left(-1 \pm \sqrt{13}\right) - \frac{4}{3} = -\frac{14}{9} \pm \frac{2\sqrt{13}}{9} = \frac{2}{9}\left(-7 \pm \sqrt{13}\right)$$

$\therefore$ There are two steady states:

$$ss1 = \left(\frac{2}{3}\left(-1+\sqrt{13}\right), \frac{2}{9}\left(-7+\sqrt{13}\right)\right) \text{ \& } \left(\frac{2}{3}\left(-1-\sqrt{13}\right), \frac{2}{9}\left(-7-\sqrt{13}\right)\right) = ss2$$

To classify these steady states, we find the eigenvalues of the Jacobian:

$$J = \begin{bmatrix} 1 & -3 \\ 2x & 4 \end{bmatrix}$$

@ ss1:

$$|J - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 1-\lambda & -3 \\ \frac{4}{3}(-1+\sqrt{13}) & 4-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda^2 - 5\lambda + 4 + 4(-1+\sqrt{13}) = 0$$

$$\Leftrightarrow \lambda^2 - 5\lambda + 4\sqrt{13} = 0$$

$$\Leftrightarrow \lambda = \frac{5 \pm \sqrt{25 - 16\sqrt{13}}}{2}$$

$\Leftrightarrow$ λ_1 & λ_2 are complex conjugates w/ +ve real part

$\therefore$ ss1 is an unstable focus.

@ ss 2:

$$|J - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 1-\lambda & -3 \\ \frac{4}{3}(-1-\sqrt{13}) & 4-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda^2 - 5\lambda + 4 + 4(-1-\sqrt{13}) = 0$$

$$\Leftrightarrow \lambda^2 - 5\lambda - 4\sqrt{13} = 0$$

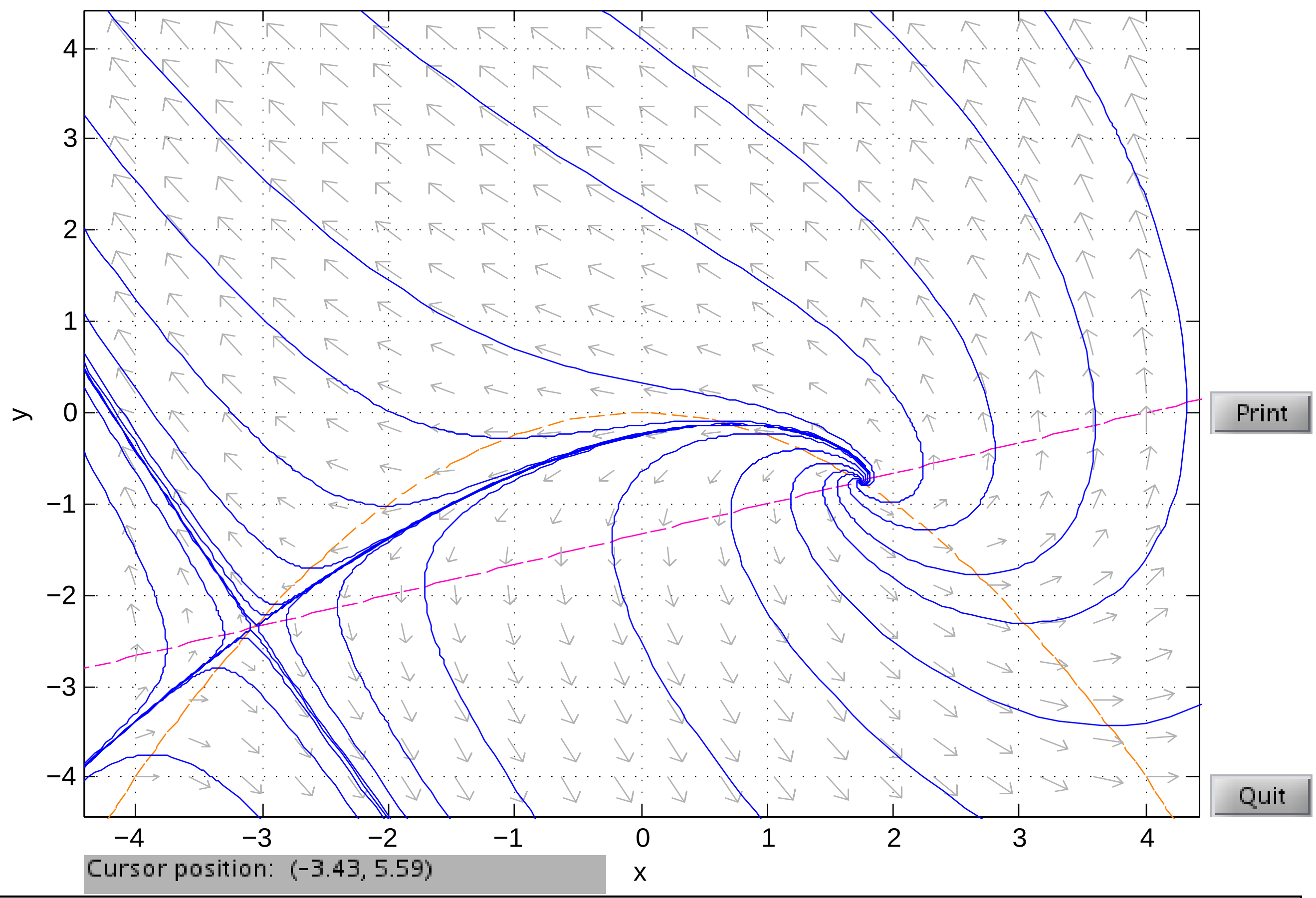
$$\Leftrightarrow \lambda = \frac{5 \pm \sqrt{25 + 16\sqrt{13}}}{2}$$

$$\Leftrightarrow \lambda_1 > 0 + \lambda_2 < 0$$

$\therefore$ ss 2 is a saddle node.

See phase plane next page.

$x' = x - 3y - 4$
$y' = x^2 + 4y$



Cursor position: (-3.43, 5.59)

The backward orbit from (-3.1, -2.5) left the computation window.
Ready.
The forward orbit from (-3, -2.4) left the computation window.
The backward orbit from (-3, -2.4) --> a possible eq. pt. near (1.7, -0.75).
Ready.