2017 - Math 339 - M1 - Sol'ns

1. a) $f(x) = x - x^2$

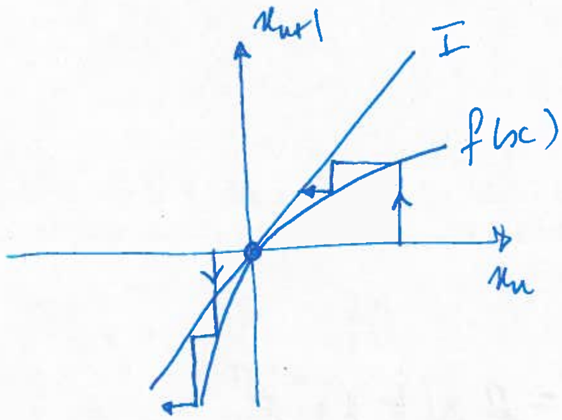
i) fixed points

$$x^* = x^* - x^{*2} \text{ for } x^* = 0 \text{ or } 1 = 1 - x^* \text{ for } x^* = 0$$

$\therefore x^* = 0$ is the only fixed point

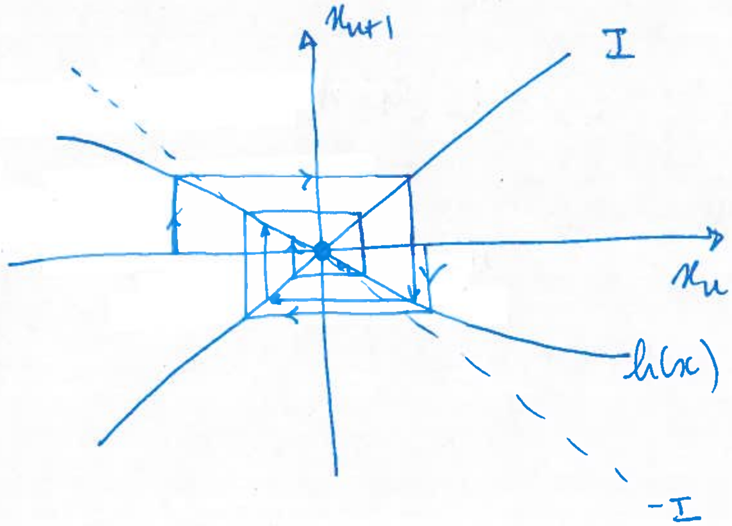
ii) $f'(x^*)\Big|_{x^*=0} = 1 - 2x^*\Big|_{x^*=0} = 1$ inconclusive

ii)



From cobwebbing, we see that $x^* = 0$ is stable for initial points $x_0 > 0$, and unstable for initial points $x_0 < 0$.

b)



We see that all initial points spiral toward the fixed point at $x^* = 0$

2. $h(x) = axe^{-x}$, $a > 0$

a) fixed points:

$x^* = ax^* e^{-x^*} \Leftrightarrow \boxed{x^* = 0}$ or $ae^{-x^*} = 1 \Leftrightarrow$

$\Leftrightarrow e^{-x^*} = \frac{1}{a}$

$\Leftrightarrow e^{x^*} = a$

$\Leftrightarrow \boxed{x^* = \ln(a)}$

$\therefore$ the two fixed points are $x^* = 0$ + $x^* = \ln(a)$.

b) $h'(x^*) = 0 \Leftrightarrow \left(ae^{-x} - axe^{-x}\right)\Big|_{x^*} = 0$

$\Leftrightarrow ae^{-\ln(a)}\left(1 - \ln(a)\right) = 0$

$\Leftrightarrow a = 0$ (not possible)

OR

$\boxed{a = e} = a_{ss}$

c) from Fig1., we see that $h'(x^*) < 0$ for $a > e$, and that it is increasing in magnitude. So we know that the steady state will be stable until

$$h'(x^*) = -1 \iff a e^{-x^*}(1-x^*) = -1$$

$$\iff a e^{-\ln(a)}(1-\ln(a)) = -1$$

$$\iff a e^{\ln(\frac{1}{a})}(1-\ln(a)) = -1$$

$$\iff \frac{a}{a}(1-\ln(a)) = -1$$

$$\iff 1-\ln(a) = -1$$

$$\iff \ln(a) = 2$$

$$\iff \boxed{a = e^2}$$

Is it possible to choose a such that

$$h'(x^*) = 1 \iff 1-\ln(a) = 1 \iff \ln(a) = 0$$

$$\iff a = 1$$

$\therefore$ the steady state $x^* = \ln(a)$ is stable for

$$\boxed{1 < a < e^2}$$

3. $f(x,y) = \left(\frac{x}{2},\ 2y - \frac{7x^3}{8}\right)$

a) finding f^{-1}:

$$\begin{cases} x_{n+1} = \frac{x_n}{2} \\ y_{n+1} = 2y_n - \frac{7x_n^3}{8} \end{cases} \Leftrightarrow \begin{cases} x_n = 2x_{n+1} \\ y_n = \frac{1}{2}\left[y_{n+1} + \frac{7}{8}(2x_{n+1})^3\right] \end{cases}$$

$$\therefore f^{-1}(x,y) = \left(2x,\ \frac{y}{2} + 7x^3\right)$$

b) $S = \{(x, rx^3) : x \in \mathbb{R}\}$

$$f(x, rx^3) = \left(2x,\ \frac{rx^3}{2} + 7x^3\right) = \left(2x, \left(\frac{r}{2}+7\right)x^3\right)$$

For S to be invariant, we require

$$r(2x)^3 = \left(\frac{r}{2}+7\right)x^3 \Leftrightarrow \forall$$

$$\forall \Leftrightarrow 8rx^3 = \left(\frac{r}{2}+7\right)x^3 \Leftrightarrow 8r = \frac{r}{2}+7$$

$$\Leftrightarrow 16r = r + 14 \Leftrightarrow 15r = 14 \Leftrightarrow r = \frac{14}{15}$$

Verify invariance under f^{-1}:

$$f^{-1}\left(x, \frac{14}{15}x^3\right) = \left(2x, \frac{14}{15}x^3\frac{1}{2} + 7x^3\right)$$

$$= \left(2x, \left(\frac{7}{15}+7\right)x^3\right)$$

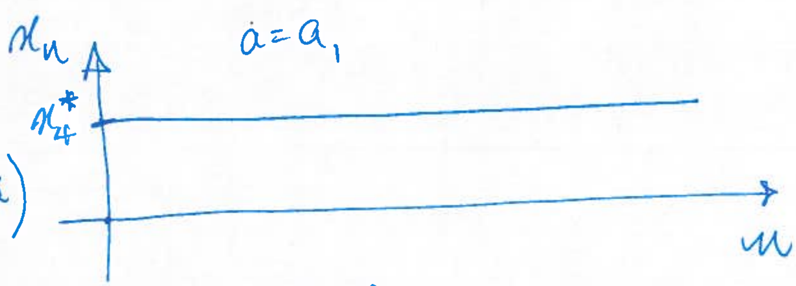
$$= \left(2x, \frac{7(1+15)}{15}x^3\right)$$

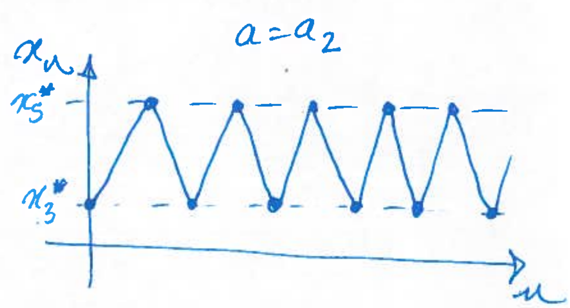
$$= \left(2x, \frac{14}{15}(8x^3)\right)$$

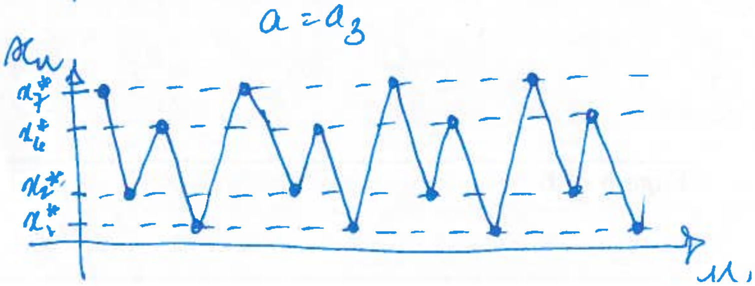
$$= \left(2x, \frac{14}{15}(2x)^3\right) \in S$$

$\therefore$ we have shown that S is invariant under iteration of f and f^{-1}!

4. a)







b) Chaos

i) sensitive dependence on initial conditions

ii) Lyapunov exponent > 0

iii) aperiodic

iv) behaviour restricted to an <u>attractor</u>

(only 3 required)