M339 - 2017

M2 - sol'ns

1. $\begin{cases} \dot{x} = 3x - y \\ \dot{y} = 2x + 4y \end{cases}$

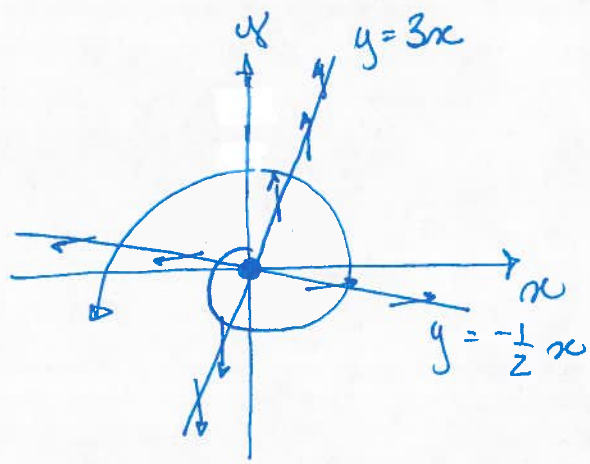
a) $J = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

$|J - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 3-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 7\lambda + 12 + 2 = 0$

$\Leftrightarrow \lambda^2 - 7\lambda + 14 = 0 \Leftrightarrow \lambda = \dfrac{7 \pm \sqrt{49 - 56}}{2}$

$\Leftrightarrow \lambda = \dfrac{7}{2} \pm i\dfrac{\sqrt{7}}{2}$

$\therefore$ the steady state is an unstable focus

b) 

$\dot{x} = 0 \Leftrightarrow y = 3x$

$\dot{y} = 0 \Leftrightarrow y = -\frac{1}{2}x$

Trajectories spiral outward in a counterclockwise direction

2. $$\begin{cases} \dot{x} = -z + \left(\frac{x^3}{3} - x\right) \\ \dot{z} = x \end{cases}$$

$V(x,z) = x^2 + z^2$

i) $V(0,0) = 0$ ✓

ii) $V(x,z) > 0 \;\forall\; (x,z) \neq (0,0)$ ✓

iii) $$\begin{aligned} \dot{V} &= 2x\dot{x} + 2z\dot{z} \\ &= 2x\left(-z + \left(\frac{x^3}{3} - x\right)\right) + 2zx \\ &= \cancel{-2xz} + 2x^2\left(\frac{x^2}{3} - 1\right) + \cancel{2xz} \\ &= 2x^2\left(\frac{x^2}{3} - 1\right) \end{aligned}$$

We require

$\dot{V} < 0 \iff \frac{x^2}{3} - 1 < 0 \iff x^2 < 3 \iff |x| < \sqrt{3}$

3. A Hamiltonian system has no dissipation, so the total energy is a Lyapunov function for the system (at any steady state) and $\dot{L} = 0$. Trajectories in the phase plane are ~~closed periodic orbits.~~ closed periodic loops.

4. $\begin{cases} \dot{x} = \rho y x - y^2 \\ \dot{y} = \dfrac{y}{x} - \rho \end{cases}$

Phase plane equ:

$$\frac{\dot{y}}{\dot{x}} = \frac{\frac{y}{x} - \rho}{\rho y x - y^2} \Leftrightarrow \frac{dy}{dx} = \frac{y - \rho x}{x} \quad \frac{1}{y(\rho x - y)}$$

$$\Leftrightarrow \frac{dy}{dx} = -\frac{1}{xy}$$

$$\Leftrightarrow y\,dy = -\frac{1}{x}\,dx$$

$$\Leftrightarrow \frac{y^2}{2} = -\ln|x| + k$$

$$\Leftrightarrow y^2 = -2\ln|x| + \tilde{k}$$

$$\Leftrightarrow y^2 = +2\ln\left|\frac{1}{x}\right| + \tilde{k}$$

$$\Leftrightarrow y^2 = \ln\left(\frac{1}{x^2}\right) + \tilde{k} \quad \ldots\ldots \quad (1)$$

The solution curves (1) are the orbits of $(x(t), y(t))$ in the phase plane, each one corresponding to a different initial condition.

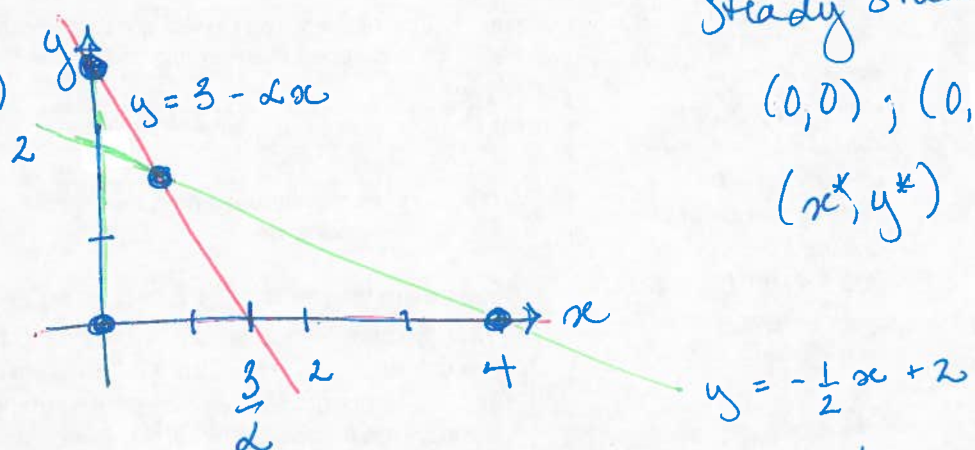
5. $\begin{cases} \dot{x} = x(4-x) - 2xy = 4x - x^2 - 2xy \\ \dot{y} = y(3-y) - \alpha xy = 3y - y^2 - \alpha xy \end{cases}$

a) $\dot{x} = 0 \iff x = 0$ or $4 - x = 2y \iff y = -\frac{1}{2}x + \frac{4}{2}$

$\iff y = -\frac{1}{2}x + 2$

$\dot{y} = 0 \iff y = 0$ or $3 - y = \alpha x \iff y = 3 - \alpha x$

b)



Steady States:

$(0,0)$; $(0,3)$; $(4,0)$; (x^*, y^*)

The coexistence steady state is given by

$4 - x = 2(3 - \alpha x) \iff 2\alpha x - x = 6 - 4$

$\iff (2\alpha - 1)x = 2$

$\iff x = \frac{2}{2\alpha - 1}$

and

$y = 3 - \alpha x = 3 - \frac{2\alpha}{2\alpha - 1} = \frac{6\alpha - 3 - 2\alpha}{2\alpha - 1}$

$= \frac{4\alpha - 3}{2\alpha - 1}$

$\because$ we require $x \geq 0$ and $y \geq 0$, the coexistence steady state

$$(x^*, y^*) = \left(\frac{2}{2\alpha - 1}, \frac{4\alpha - 3}{2\alpha - 1} \right)$$

only exists when $\alpha > \frac{1}{2}$ or $\boxed{\alpha > \frac{3}{4}}$. We choose the larger of the two.

c) Stability

$$J = \begin{bmatrix} 4 - 2x - 2y & -2x \\ -\alpha y & 3 - 2y - \alpha x \end{bmatrix}$$

at $(0,0)$

$$J\Big|_{(0,0)} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

$\therefore \lambda_1 = 4,\ \lambda_2 = 3$ and this steady state is an unstable node.

at $(0,3)$

$$J\Big|_{(0,3)} = \begin{bmatrix} 4 - 6 & 0 \\ -3\alpha & 3 - 6 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -2\alpha & -3 \end{bmatrix}$$

$\therefore \lambda_1 = -2,\ \lambda_2 = -3$, and this steady state is a stable node.

at $(4,0)$

$$J\Big|_{(4,0)} = \begin{bmatrix} 4-8 & -8 \\ 0 & 3-4\alpha \end{bmatrix} = \begin{bmatrix} -4 & -8 \\ 0 & 3-4\alpha \end{bmatrix}$$

$\therefore \lambda_1 = -4, \quad \lambda_2 = 3-4\alpha$

If $3-4\alpha < 0 \iff \alpha > \frac{3}{4}$, thus steady state is a stable node.

If $3-4\alpha > 0 \iff \alpha < \frac{3}{4}$, this steady state is a saddle(node).

at $\left(\frac{2}{2\alpha-1}, \frac{4\alpha-3}{2\alpha-1} \right)$

$$J\Big|_{*} = \begin{bmatrix} 4 - \frac{4}{2\alpha-1} - \frac{2(4\alpha-3)}{2\alpha-1} & \frac{-4}{2\alpha-1} \\ -\frac{\alpha(4\alpha-3)}{2\alpha-1} & 3 - \frac{2(4\alpha-3)}{2\alpha-1} - \frac{2\alpha}{2\alpha-1} \end{bmatrix}$$

$$= \frac{1}{2\alpha-1} \begin{bmatrix} 8\alpha-4-4-8\alpha+6 & -4 \\ -\alpha(4\alpha-3) & 6\alpha-3-2(4\alpha-3)-2\alpha \end{bmatrix}$$

$$= \frac{1}{2\alpha-1} \begin{bmatrix} -2 & -4 \\ -\alpha(4\alpha-3) & 15-20\alpha \end{bmatrix}$$

$$= \frac{1}{2\alpha-1} \underbrace{\begin{bmatrix} -2 & -4 \\ -\alpha(4\alpha-3) & 3-4\alpha \end{bmatrix}}$$

where $\begin{cases} 2\alpha-1 > 0 \\ 4\alpha-3 > 0 \end{cases}$

Let $\eta = 4\alpha - 3$. We know that $\alpha > \frac{3}{4}$ so $\eta > 0$.
Then $\alpha = \frac{1}{4}(\eta+3)$ and we can rewrite J:

$$J = \begin{bmatrix} -2 & -4 \\ -\frac{1}{4}(\eta+3)\eta & -\eta \end{bmatrix}$$

The eigenvalues of J are given by

$$|J - \lambda I| = 0 \iff \begin{vmatrix} -2-\lambda & -4 \\ -\frac{1}{4}(\eta+3)\eta & -\eta-\lambda \end{vmatrix} = 0$$

$$\iff \lambda^2 + (2+\eta)\lambda + 2\eta - (\eta+3)\eta = 0$$

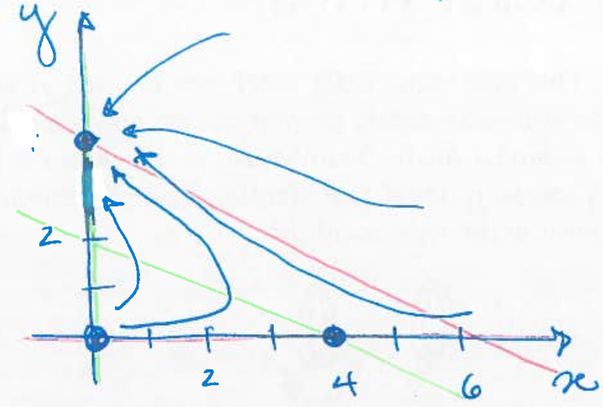
$$\iff \lambda^2 + (2+\eta)\lambda + \eta(2-\eta-3) = 0$$

$$\iff \lambda^2 + (2+\eta)\lambda - \eta(\eta+1) = 0$$

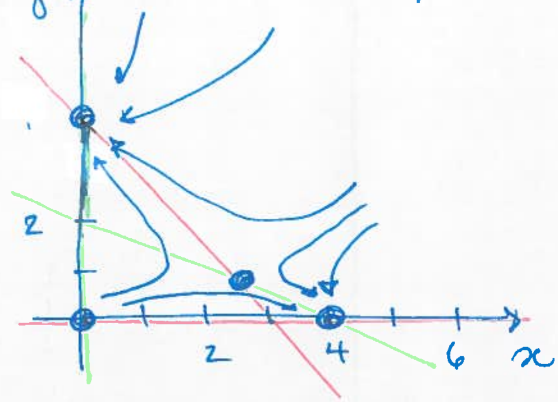
$$\iff \lambda = \frac{-(2+\eta) \pm \sqrt{(2+\eta)^2 + 4\eta(\eta+1)}}{2}$$

$\therefore$ $\lambda_1 > 0$ and $\lambda_2 < 0$, and so this steady state is a saddle (node) d)

case 1: $\alpha < \frac{3}{4}$



case 2: $\alpha > \frac{3}{4}$



e) Bifurcation diagram