A#1 - Math 339, 2019

Solutions

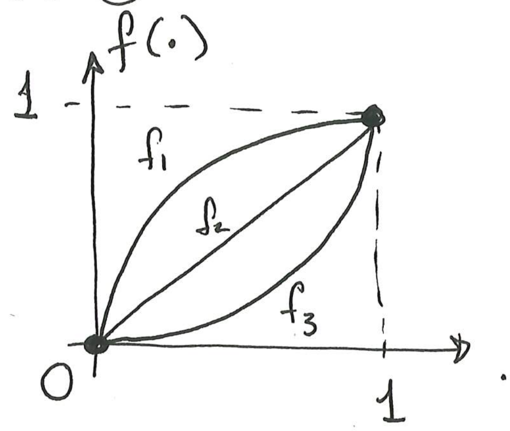
Q1)

focal indiv. \ neighbour	S_{-1}	X_{-1}	X_{+1}	S_{+1}
S_{-1}	S_{-1}	S_{-1}	S_{-1} or X_{-1}	S_{-1}
X_{-1}	X_{-1} or S_{-1}	X_{-1}	X_{+1}	X_{+1}
X_{+1}	X_{-1}	X_{-1}	X_{+1}	X_{+1} or S_{+1}
S_{+1}	S_{+1}	S_{+1} or X_{+1}	S_{+1}	S_{+1}

The rates of switching in the ambiguous cases are determined by the function $f(\cdot)$:

$$\begin{tikzcd}
\boxed{S_{-1}} \arrow[r, bend left, "f(X_{+1})"] & \boxed{X_{-1}} \arrow[r, bend left, "1"] \arrow[l, bend left, "1-f(S_{-1})"] & \boxed{X_{+1}} \arrow[r, bend left, "1-f(S_{+1})"] \arrow[l, bend left, "1"] & \boxed{S_{+1}} \arrow[l, bend left, "f(X_{-1})"]
\end{tikzcd}$$

a) We know that $0 \le f(\cdot) \le 1$. We assume that $f(0)=0$ and $f(1)=1$. So any function satisfying these 3 criteria is possible.



Some options are sketched at left.

f_1 means it takes few individuals to make a switch likely

f_3 means that it takes a lot of individuals to make a switch likely

f_2 is in between

b) Below, I use the convention that in each interaction the focal individual appears first, & the neighbour appears second. I also write each $(\dot{\ })$ as gain interactions less loss interactions.

$$
\begin{cases}
\dfrac{dS_{-1}}{dt} = \left(1-f(S_{-1})\right)X_{-1}S_{-1} - f(X_{+1})\,S_{-1}X_{+1} \\[2ex]
\dfrac{dX_{-1}}{dt} = f(X_{+1})\,S_{-1}X_{+1} + X_{+1}(S_{-1}+X_{-1}) \\
\qquad\qquad - \left(1-f(S_{-1})\right)X_{-1}S_{-1} - X_{-1}(X_{+1}+S_{+1}) \\[2ex]
\dfrac{dX_{+1}}{dt} = X_{-1}(X_{+1}+S_{+1}) + f(X_{-1})\,S_{+1}X_{-1} \\
\qquad\qquad - X_{+1}(S_{-1}+X_{-1}) - \left(1-f(S_{+1})\right)X_{+1}S_{+1} \\[2ex]
\dfrac{dS_{+1}}{dt} = \left(1-f(S_{+1})\right)X_{+1}S_{+1} - f(X_{-1})\,S_{+1}X_{-1}
\end{cases}
$$

Q2)

	L_2	L_1	R_1	R_2
L_2	L_2	L_1 or L_2	L_1	L_1
L_1	L_2	L_1 or L_2	R_1	R_1
R_1	L_1	L_1	R_1 or R_2	R_2
R_2	R_1	R_1	R_1 or R_2	R_2

$$\frac{dL_1}{dt} = [\text{gains}] - [\text{losses}]$$

$$= \left[(1-pa)L_2L_1 + L_2(R_1+R_2) + R_1(L_2+L_1)\right]$$

$$- \left[paL_1L_1 + L_1(L_2+R_1+R_2)\right] \quad \cdots (1)$$

OR $$= [\text{gains}] - \left(L_1 - [\text{no change}]\right)$$

$$= \left[(1-pa)L_2L_1 + L_2(R_1+R_2) + R_1(L_2+L_1)\right]$$

$$- \underbrace{\left(L_1 - \left[(1-pa)L_1^2\right]\right)}_{1} \quad \cdots (2)$$

In the paper we have

$$\mathring{L}_1 = L_2[1-L_2-p_a L_1] + R_1[L_2+L_1] \underbrace{-L_1[1-(1-p_a)L_1]}_{1} \quad \ldots\ldots (3)$$

We observe that terms 1 are the same, so we compare equation (3) with equation (2). Rearranging the gain terms in equation (2), we have

$$\begin{aligned} G_2 &= (1-p_a)L_2L_1 + L_2(R_1+R_2) + R_1(L_2+L_1) \\ &= L_2\left[(1-p_a)L_1 + R_1+R_2+R_1\right] + R_1L_1 \\ &= L_2\left[L_1+R_1+R_2 - p_aL_1 + R_1\right] + R_1L_1 \end{aligned} \quad \ldots\ldots (4)$$

Note that $L_2+L_1+R_1+R_2 = 1 \Leftrightarrow$ //

$$// \Leftrightarrow L_1+R_1+R_2 = 1-L_2 \quad \ldots\ldots (5)$$

Plugging (5) into (4) we obtain

$$G_2 = L_2[1 - L_2 - p_a L_1] + R_1 L_2 + R_1 L_1$$

$$= L_2[1 - L_2 - p_a L_1] + R_1[L_1 + L_2]$$

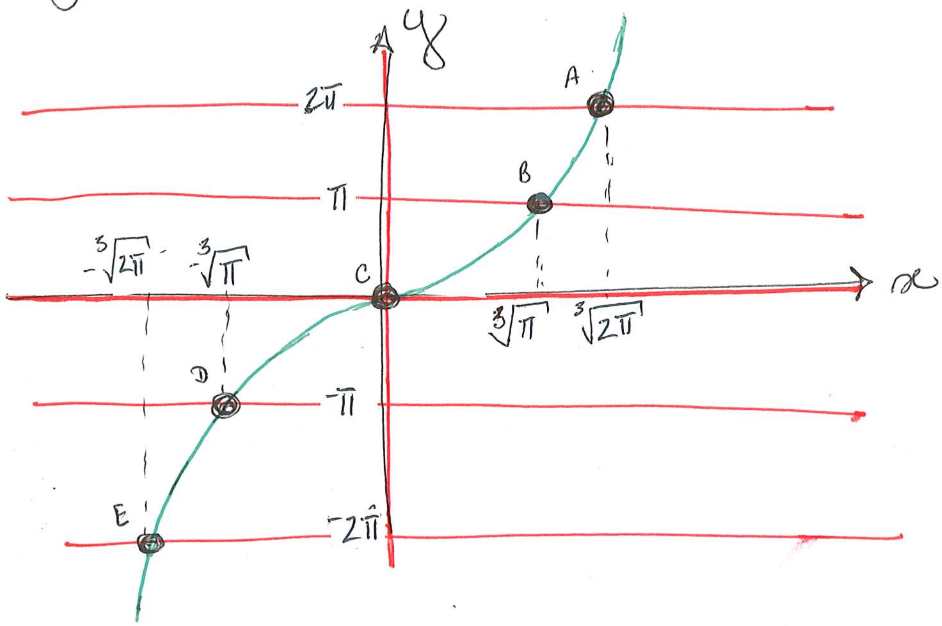
which ~~is~~ is equal to the gain term in equation (3), as required.

3. a) $\begin{cases} \dot{x} = x\sin(y) = f(x,y) \\ \dot{y} = x^3 - y = g(x,y) \end{cases}$

Nullclines:

$\dot{x} = 0 \iff x\sin(y) = 0 \iff \underline{x = 0}$ or $\underline{y = n\pi}$, $n \in \mathbb{Z}$

$\dot{y} = 0 \iff x^3 - y = 0 \iff \underline{y = x^3}$



In the region $-7 < y < 7$, there are five steady states (A – E):

$(\sqrt[3]{2\pi}, 2\pi)$; $(\sqrt[3]{\pi}, \pi)$; $(0,0)$;

$(-\sqrt[3]{\pi}, -\pi)$; $(-\sqrt[3]{2\pi}, -2\pi)$

Next, we determine the vector field around each steady state.

Consider first the y-nullcline, and start from the top right: To determine the horizontal flow on the y-nullcline, we evaluate $f(x,y)$ at relevant points:

A. $f(\sqrt[3]{2\pi}+\varepsilon,\ 2\pi+\varepsilon) = (\sqrt[3]{2\pi}+\varepsilon)\sin(2\pi+\varepsilon) > 0$

$\therefore$ flow is $\rightarrow$

$f(\sqrt[3]{2\pi}-\varepsilon,\ 2\pi-\varepsilon) = (\sqrt[3]{2\pi}-\varepsilon)\sin(2\pi-\varepsilon) < 0$

$\therefore$ flow is $\leftarrow$

In general, we expect the flow to continue flipping in this manner each time a steady state is crossed.

B. So flow at $(\sqrt[3]{\pi}+\varepsilon,\ \pi+\varepsilon)$ is $\leftarrow$ b/c no steady state was crossed between this point and the previous one.

We expect flow at $(\sqrt[3]{\pi}-\varepsilon,\ \pi-\varepsilon)$ to be $\rightarrow$. Let's check:

$f(\sqrt[3]{\pi}-\varepsilon,\ \pi-\varepsilon) = (\sqrt[3]{\pi}-\varepsilon)\sin(\pi-\varepsilon) > 0$ ✓

C. At $(0,0)$ however, since there are two x-nullclines that pass through the steady state, the flipping pattern may change.

$$f(0+\varepsilon, 0+\varepsilon) = (0+\varepsilon)\sin(0+\varepsilon) \sim \varepsilon^2 > 0 \therefore \rightarrow$$

$$f(0-\varepsilon, 0-\varepsilon) = (0-\varepsilon)\sin(0-\varepsilon) \sim (-\varepsilon)^2 > 0 \therefore \rightarrow$$

So flow does not flip across that steady state. We now check the last two crossings:

D. $$f(-\sqrt[3]{\pi} - \varepsilon, -\pi - \varepsilon) = \underbrace{(-\sqrt[3]{\pi} - \varepsilon)}_{<0}\underbrace{\sin(-\pi - \varepsilon)}_{>0} < 0 \therefore \leftarrow$$

E. $$f(-\sqrt[3]{2\pi} - \varepsilon, -2\pi - \varepsilon) = \underbrace{(-\sqrt[3]{2\pi} - \varepsilon)}_{<0}\underbrace{\sin(-2\pi - \varepsilon)}_{<0} > 0 \therefore \rightarrow$$

So the flow flips in the usual manner at these two crossings.

Now consider the x-nullclines, again starting from the steady state at the top right. Flow direction across the x-nullcline is vertical & so we evaluate $g(x,y)$ at relevant points:

A. $g(\sqrt[3]{2\pi}+\varepsilon, 2\pi) = (\sqrt[3]{2\pi}+\varepsilon)^3 - 2\pi > 0 \therefore \uparrow$

$g(\sqrt[3]{2\pi}-\varepsilon, 2\pi) = (\sqrt[3]{2\pi}-\varepsilon)^3 - 2\pi < 0 \therefore \downarrow$

B. This same pattern will hold at the next steady state.

C. Consider now the $(0,0)$ steady state:

$g(\varepsilon, 0) = \varepsilon^3 - 0 > 0 \therefore \uparrow$

$g(-\varepsilon, 0) = (-\varepsilon)^3 - 0 < 0 \therefore \downarrow$

$g(0, \varepsilon) = 0 - \varepsilon < 0 \therefore \downarrow$

$g(0, -\varepsilon) = 0 - (-\varepsilon) > 0 \therefore \uparrow$

D. Consider now the $(\sqrt[3]{-\pi}, -\pi)$ steady state:

$g(\sqrt[3]{-\pi}+\varepsilon, -\pi) = (\sqrt[3]{-\pi}+\varepsilon)^3 - (-\pi)$

$= \cancel{-\pi} + (\pi)^{2/3}\varepsilon + \cancel{\pi} + O(\varepsilon^2)$

$= +(\pi)^{2/3}\varepsilon + O(\varepsilon^2) > 0 \therefore \uparrow$

~~E. Finally, consider the last steady state:~~

$$\cancel{g\left(\sqrt[3]{-2\pi} - \varepsilon, -2\pi\right) = \left(\sqrt[3]{-2\pi} - \varepsilon\right)^3 - (-2\pi)}$$

$$\cancel{= -2\pi + 2\pi - 3\left(\sqrt[3]{-2\pi}\right)^2 \varepsilon + O(\varepsilon^2)}$$

$$\cancel{= \ldots}$$

sorry!

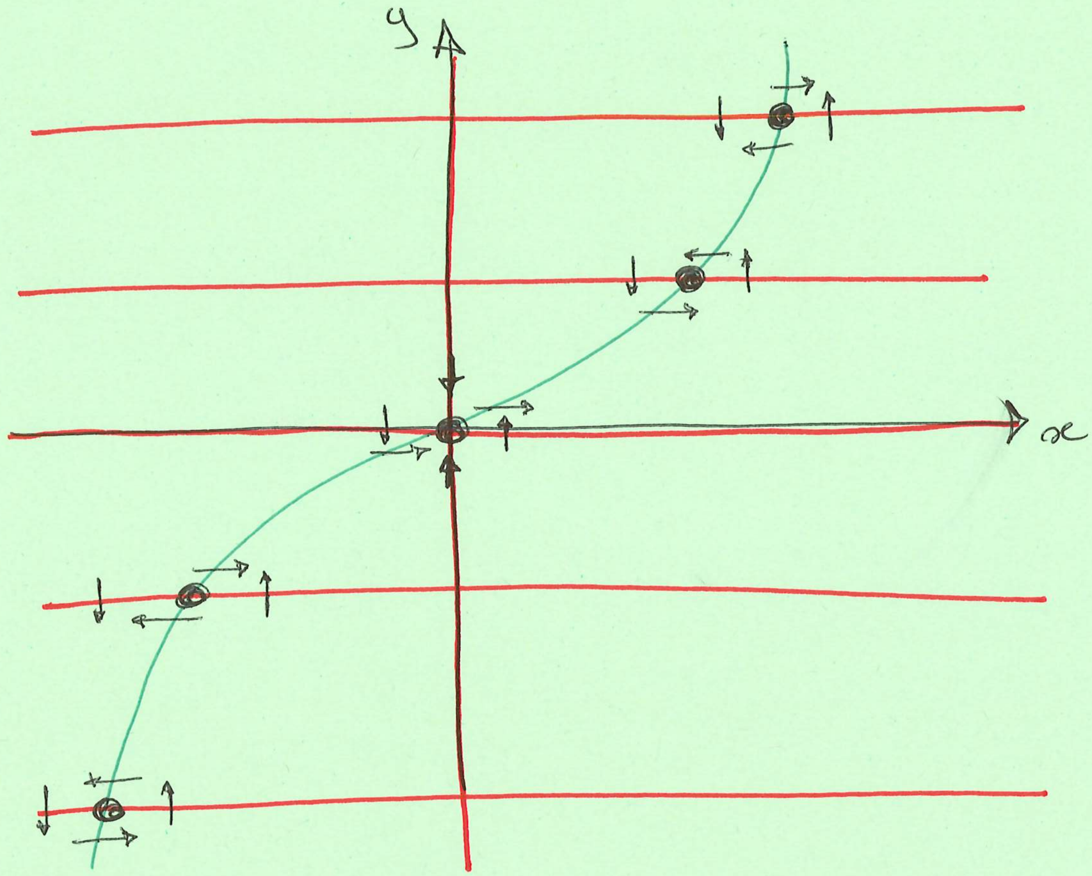
$$g\left(\sqrt[3]{-\pi} - \varepsilon, -\pi\right) = \left(\sqrt[3]{-\pi} - \varepsilon\right)^3 - (-\pi)$$

$$= -\pi - (\pi)^{2/3}\varepsilon + \pi + O(\varepsilon^2)$$

$$= -(\pi)^{2/3}\varepsilon + O(\varepsilon^2) < 0 \therefore \downarrow$$

E. This same pattern will hold at the next steady state.

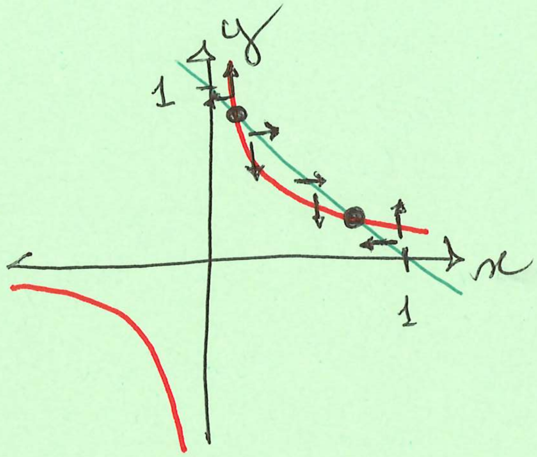
We can now complete the phase plane diagram (next p.).


y
x

b) $\begin{cases} \dot{x} = 10xy - 1 & = f(x,y) \\ \dot{y} = x + y - 1 & = g(x,y) \end{cases}$

Nullclines:

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} 10xy - 1 = 0 \\ x + y - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} y = \frac{1}{10x} \\ y = 1 - x \end{cases}$$



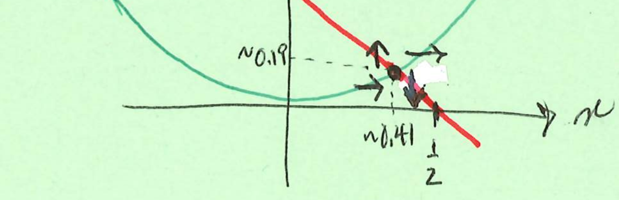
Two steady states, $\left(\frac{5 \pm \sqrt{15}}{10}, \frac{5 \mp \sqrt{15}}{10}\right)$

Flow: as shown

c) $\begin{cases} \dot{x} = (y + 2x - 1)^2 & = f(x,y) \\ \dot{y} = y - x^2 & = g(x,y) \end{cases}$

Nullclines:

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} y + 2x - 1 = 0 \\ y - x^2 = 0 \end{cases} \Leftrightarrow \begin{cases} y = 1 - 2x \\ y = x^2 \end{cases}$$



Two steady states.

Flow: as shown

Note that $f(x,y) \geq 0 \ \forall (x,y)$, so the flow in x is always $\rightarrow$.