Math 339 - Sep-Dec 2019
Assignment #2 - Solutions

1. a) $\begin{cases} \dot{x} = x\sin(y) = f(x,y) \\ \dot{y} = x^3 - y = g(x,y) \end{cases}$ $(x^*, y^*) = (\sqrt[3]{2\pi}, 2\pi)$

We linearise about the steady state $*$:

$$A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} \sin(y) & x\cos(y) \\ 3x^2 & -1 \end{bmatrix}$$

At the steady-state (or rather, near the steady state), we have the linear system

$$\dot{\vec{x}} = \begin{bmatrix} \sin(2\pi) & \sqrt[3]{2\pi}\cos(2\pi) \\ 3(\sqrt[3]{2\pi})^2 & -1 \end{bmatrix} \vec{x}$$

$$= \begin{bmatrix} 0 & (2\pi)^{1/3} \\ 3(2\pi)^{2/3} & -1 \end{bmatrix} \vec{x} = A|_* \vec{x}$$

Stability is determined by the eigenvalues of the coefficient matrix. To find the eigenvalues, we solve

$$|A - \lambda I| = 0 \iff \begin{vmatrix} -\lambda & (2\pi)^{1/3} \\ 3(2\pi)^{2/3} & -1-\lambda \end{vmatrix} = 0$$

$$\iff \lambda^2 + \lambda - 3(2\pi) = 0$$

$$\iff \lambda = \frac{-1 \pm \sqrt{1 + 24\pi}}{2}$$

$\therefore$ λ_1 + λ_2 are both real, with one eigenvalue positive + the other negative.

So the steady state is a saddle.

We could have also seen this result by observing:

$$\begin{cases} Tr(A|_*) = -1 < 0 \\ Det(A|_*) = -6\pi < 0 \end{cases}$$

$\therefore$ we know that the steady state must be a saddle.

b) $\begin{cases} \dot{x} = 10xy - 1 \\ \dot{y} = x + y - 1 \end{cases}$ $\quad (x^*, y^*) = \left(\frac{5+\sqrt{15}}{10}, \frac{5-\sqrt{15}}{10} \right)$

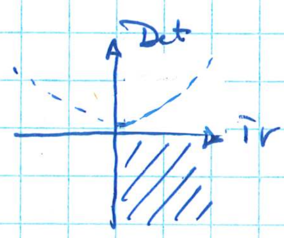
Linearising about the steady state we obtain

$$A = \left.\begin{bmatrix} 10y & 10x \\ 1 & 1 \end{bmatrix}\right|_* = \begin{bmatrix} 5-\sqrt{15} & 5+\sqrt{15} \\ 1 & 1 \end{bmatrix}$$

Observe:

$Tr(A) = 6 - \sqrt{15} > 0$

$Det(A) = 5 - \sqrt{15} - 5 - \sqrt{15} = -2\sqrt{15} < 0$



So the steady state is a saddle.

Check: Solve

$|A - \lambda I| = 0 \iff \lambda^2 - (6 - \sqrt{15})\lambda - 2\sqrt{15} = 0$

$\iff \lambda = \frac{(6-\sqrt{15}) \pm \sqrt{16 - 8\sqrt{15} + 15 + 8\sqrt{15}}}{2}$

$= \frac{(6-\sqrt{15}) \pm \sqrt{31}}{2} \Rightarrow \lambda_+ > 0, \lambda_- < 0$ ✓

c) $\begin{cases} \dot{x} = (y + 2x - 1)^2 \\ \dot{y} = y - x^2 \end{cases}$ $\qquad (x^*, y^*) = (-1 - \sqrt{2}, 3 + 2\sqrt{2})$

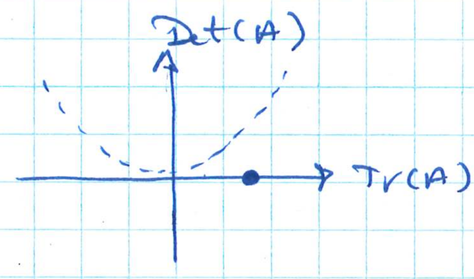
Linearising about the steady state we obtain

$$A = \left.\begin{bmatrix} 2(y+2x-1)2 & 2(y+2x-1) \\ -2x & 1 \end{bmatrix}\right|_{*}$$

$$= \begin{bmatrix} 4(3+2\sqrt{2}-2-2\sqrt{2}-1) & 2(3+2\sqrt{2}-2-2\sqrt{2}-1) \\ -2(-1-\sqrt{2}) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 2+2\sqrt{2} & 1 \end{bmatrix}$$

$\therefore \begin{cases} Tr(A) = 1 > 0 \\ Det(A) = 0 \end{cases}$



Inconclusive! Consider the eigenvalues:

$$|A - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} -\lambda & 0 \\ 2+2\sqrt{2} & 1-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda^2 - \lambda = 0 \Leftrightarrow \lambda(1-\lambda) = 0$$

$$\Leftrightarrow \lambda_1 = 0 \ \& \ \lambda_2 = 1$$

neutrally stable $\qquad$ unstable

Since one eigenvalue is positive, ~~the~~ steady state is unstable. We cannot give the steady state a name however, b/c it lies on the boundary between two classifications.

2. $\begin{cases} \dot{x} = 3x - y \\ \dot{y} = 6x - 4y \end{cases}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

The line defined by $\vec{v}_2$ is

$$L : y - 6x = 0$$

To show that L is an invariant set for the system we compute

$$\begin{aligned} \dot{L} &= 6x - 4y - 6(3x - y) \\ &= 6x - 4y - 18x + 6y \\ &= -12x + 2y \\ &= 2(y - 6x) \\ &= 0 \quad (\text{for } y - 6x = 0) \end{aligned}$$

$\therefore$ L is an invariant set for the dynamical system.

$\because$ solutions are of the form $\vec{x} = c_1\vec{v}_1 e^{2t} + c_2\vec{v}_2 e^{-3t}$, we know that solution trajectories that start on L remain on L & get closer to $(0,0)$ $\therefore$ L is a stable manifold for the dynamical system.

Since the eigenvalue associated with $\vec{v}_2$ is negative.

3. $$\begin{cases} \dot{x} = -y^2 - [(p_a - 1)x - 1]y + [(2 - p_a)x - 1]x \\ \dot{y} = y^2 + [(p_a + 1)x - 1]y + p_a x^2 \end{cases}$$

$$L = \frac{1}{2} - x - y = 0$$

$L = 0$ is an invariant set for the system if $L = 0 \Rightarrow \dot{L} = 0$.
We compute

$$\begin{aligned} \dot{L} &= -\dot{x} - \dot{y} \\ &= \cancel{y^2} + [(p_a - 1)x - \cancel{1}]y - [(2 - p_a)x - 1]x \\ &\quad - \cancel{y^2} - [(p_a + 1)x - \cancel{1}]y - p_a x^2 \\ &= -2xy - 2x^2 + x \\ &= x(-2y - 2x + 1) \\ &= 2x\underbrace{\left(\frac{1}{2} - x - y\right)}_{= 0 \ \because L = 0} \\ &= 0 \end{aligned}$$

$\therefore$ the set $L: \frac{1}{2} - x - y = 0$ is an invariant set for the dynamical system, $\forall\, p_a \in [0, 1]$

Stability is determined from the eigenvalues. When $p_a = 0$ we have

$$\begin{cases} \dot{x} = -y^2 + (x+1)y + (2x-1)x \\ \dot{y} = y^2 + (x-1)y \end{cases}$$

$$\therefore A = \begin{bmatrix} y + 4x - 1 & x + 1 \\ y & 2y + x - 1 \end{bmatrix}$$

Now we need to determine the coordinates of the steady state:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Leftrightarrow \begin{cases} -y^2 + (x+1)y + (2x-1)x = 0 \\ y^2 + (x-1)y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2xy + (2x-1)x = 0 \\ y^2 + (x-1)y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x^2 + (2y-1)x = 0 \\ y = 0 \quad \text{or} \quad y + (x-1) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \quad \text{or} \quad 2x + (2y-1) = 0 \\ y = 0 \quad \text{or} \quad y = 1-x \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \quad \text{or} \quad y = \frac{1}{2} - x \\ y = 0 \quad \text{or} \quad y = 1-x \end{cases}$$

So the steady states are

$$(0,0)\,;\; \left(\tfrac{1}{2}, 0\right);\; (0,1)$$

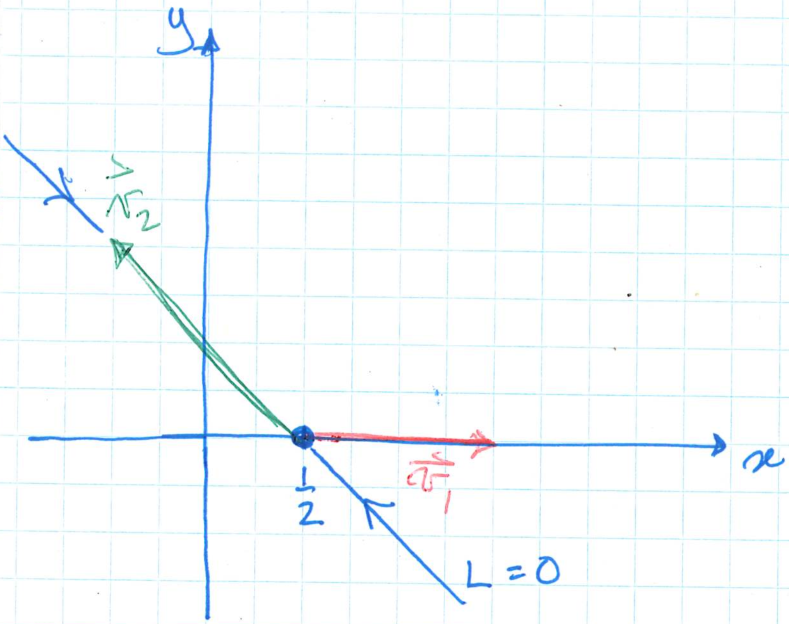
The only steady states that is on the $L = \frac{1}{2} - x - y = 0$ manifold is $\left(\frac{1}{2}, 0\right)$. So we compute the eigenvalues of A at this steady state:

$$A\Big|_{(\frac{1}{2},0)} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} \quad \therefore \quad \begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -\frac{1}{2} \end{aligned}$$

and we see that $\left(\frac{1}{2}, 0\right)$ is a saddle. Now we need the eigenvectors associated with the two eigenvalues:

$\lambda_1 = 1$: $[A - \lambda I]\vec{v}_1 = \vec{0} \Leftrightarrow \begin{bmatrix} 0 & \frac{3}{2} \\ 0 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -\frac{1}{2}$: $[A - \lambda I]\vec{v}_2 = \vec{0} \Leftrightarrow \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ choose $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



$\therefore$ $\vec{v}_2$ points along $L=0$, the eigenvalue associated with $\vec{v}_2$ is $\lambda_2 = -\frac{1}{2} < 0$, and so the $L=0$ invariant set is a stable manifold for the $(\frac{1}{2}, 0)$ steady state.

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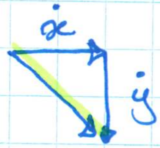
Alternatively, we could determine direction of motion along $\vec{v}_2$ by plugging a point on $L=0$ into the dynamical system. Take, for example, the point $(0, \frac{1}{2})$.

$$\dot{x}\Big|_{(0,\frac{1}{2})} = \left[-y^2 + (x+1)y + (2x-1)x\right]\Big|_{(0,\frac{1}{2})} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

$$\dot{y}\Big|_{(0,\frac{1}{2})} = \left[y^2 + (x-1)y\right]\Big|_{(0,\frac{1}{2})} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

$\therefore$ the flow direction is 

which is toward $(\frac{1}{2}, 0)$, indicating that $L=0$ is a stable manifold for this steady state.