Math 339 - Dynamical Systems
Sep-Dec 2019
Assignment # 3

Instructions: You are being evaluated on the presentation, as well as the correctness, of your answers. Try to answer questions in a clear, direct, and efficient way. Sloppy or incorrect use of technical terms will lower your mark.

1. Consider the nonlinear system

$$\dot{x} = x - xy - y + y^2, \tag{1}$$
$$\dot{y} = x^2 - y. \tag{2}$$

For this dynamical system, do the full phase plane analysis. That is:

(a) Sketch the nullclines and find the steady states (use colour!).

(b) Linearise the system around each of the steady states to determine their stability and classify them.

(c) For the saddles, find the eigenvectors.

(d) Draw flow direction arrows on the nullclines and eigenvectors.

(e) Use all of the information gathered above to sketch possible solutions in the full phase plane. *(Note: The reason we can construct the global behaviour just from the local information at each steady state is a characteristic of two dimensional systems. We will be discussing this feature in class on Thursday.)*

2. In class today, I stated that 2×2 flows have special attributes not shared by flows in higher dimensions. In this problem, we explore one of these distinctions.

(a) In diagram (a) of Figure 1 (next page), a closed orbit has been drawn in the xy plane. The arrows **A** and **B** represent the local directions of motion at two points on the inside and outside of the closed curve.

i. By preserving a continuous flow, sketch several *different* qualitative flow patterns consistent with the diagram.

ii. In class, I mentioned that a closed curve in the plane must contain at least one steady state must be inside it. Verify that this statement makes sense with regard to each of the different flow patterns you have drawn.

(b) A similar diagram in three dimensions (for a system xyz of three equations) is shown in diagram (b) of Figure 1. This diagram leads to some ambiguity. Is it possible to define inside and outside regions for the orbit? Give some sketches or verbal descriptions of flow patterns consistent with this orbit. Show that it is not necessary to assume that a steady state is associated with the closed orbit.

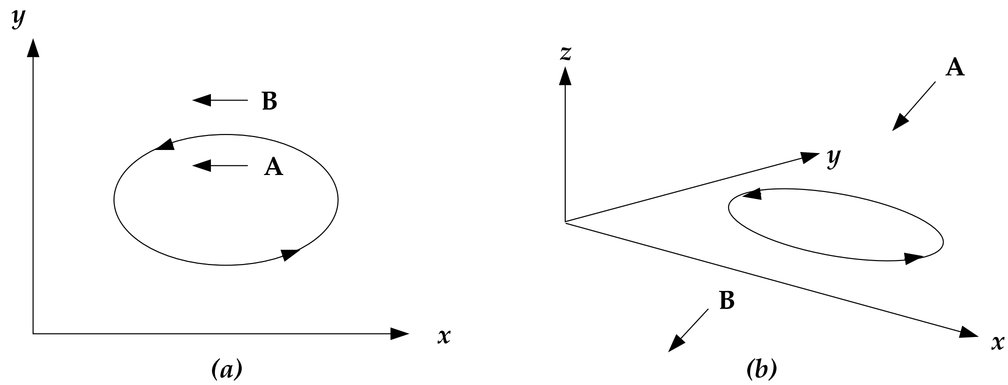

Figure 1: Exploring flows in 2 and 3 dimensions.

3. Using the scaling ratios given in class, show that equations (2) and (3) (see lecture notes) are the same model.

4. In class we saw a transcritical bifurcation, in which there was an exchange of stability between two steady states. There are several other types of bifurcations, a few of which you will explore in this problem.

   Let x be the state variable and $\mu \in \mathbb{R}$ a bifurcation parameter. Sketch bifurcation diagrams for the differential equations listed below. Note that the bifurcation diagram is a plot of the steady states x^* as functions of the bifurcation parameter μ. In each case, explains what happens as μ increases from values below to values above the bifurcation point.

   (a) saddle-node bifurcation: $\dot{x} = \mu - x^2$
   (b) transcritical bifurcation: $\dot{x} = \mu x - x^2$
   (c) supercritical pitchfork bifurcation: $\dot{x} = \mu x - x^3$
   (d) subcritical pitchfork bifurcation: $\dot{x} = \mu x + x^3$