Math 339 - Sep-Dec 2019
A#3 - Solutions

1. $\begin{cases} \dot{x} = x - xy - y + y^2 = f(x,y) \\ \dot{y} = x^2 - y = g(x,y) \end{cases}$

a) Nullclines

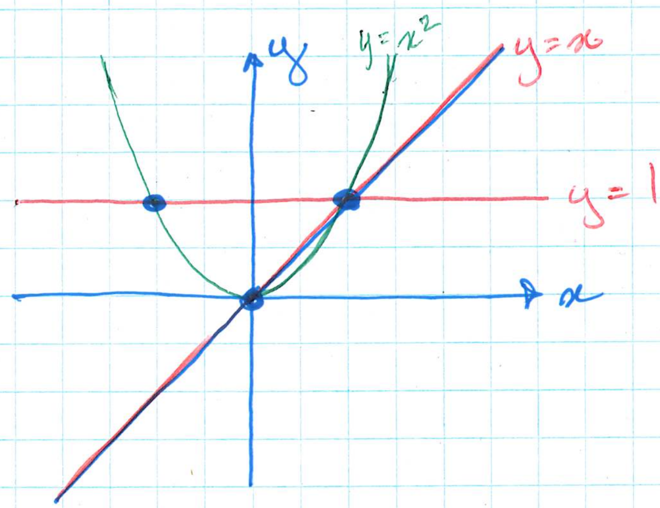
$f(x,y)=0 \iff x - xy - y + y^2 = 0$

$\iff x(1-y) - y(1-y) = 0$

$\iff (x-y)(1-y) = 0$

$\iff y = x$ or $y = 1$

$g(x,y)=0 \iff x^2 - y = 0 \iff y = x^2$



$\therefore$ Steady states are

$(-1, 1)$
$(0, 0)$
$(1, 1)$

b) The linearized system is $\dot{\vec{x}} = A\vec{x}$ where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$.

We have

$$f_x = 1 - y \qquad f_y = -x - 1 + 2y$$

$$g_x = 2x \qquad g_y = -1$$

$$\therefore A = \begin{bmatrix} 1-y & 2y - x - 1 \\ 2x & -1 \end{bmatrix}$$

<u>At (0,0)</u>

$$A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$ $\therefore$ $\lambda_1 = 1$, $\lambda_2 = -1$ and $(0,0)$ is a $\boxed{\text{saddle node}}$

eigenvectors:

<u>$\lambda_1 = 1$</u> (unstable direction)

$$[A - \lambda I]\vec{v}_1 = \vec{0} \Leftrightarrow \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can $\therefore$ choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

<u>$\lambda_2 = -1$</u> (stable direction)

$$[A - \lambda_2 I]\vec{v}_2 = \vec{0} \Leftrightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can choose $\therefore$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

At $(-1,1)$

$$A = \begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix}$$

eigenvalues satisfy

$$-\lambda(-1-\lambda) + 4 = 0 \iff \lambda^2 + \lambda + 4 = 0 \iff$$

$$\iff \lambda = \frac{-1 \pm \sqrt{1-16}}{2} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2} i$$

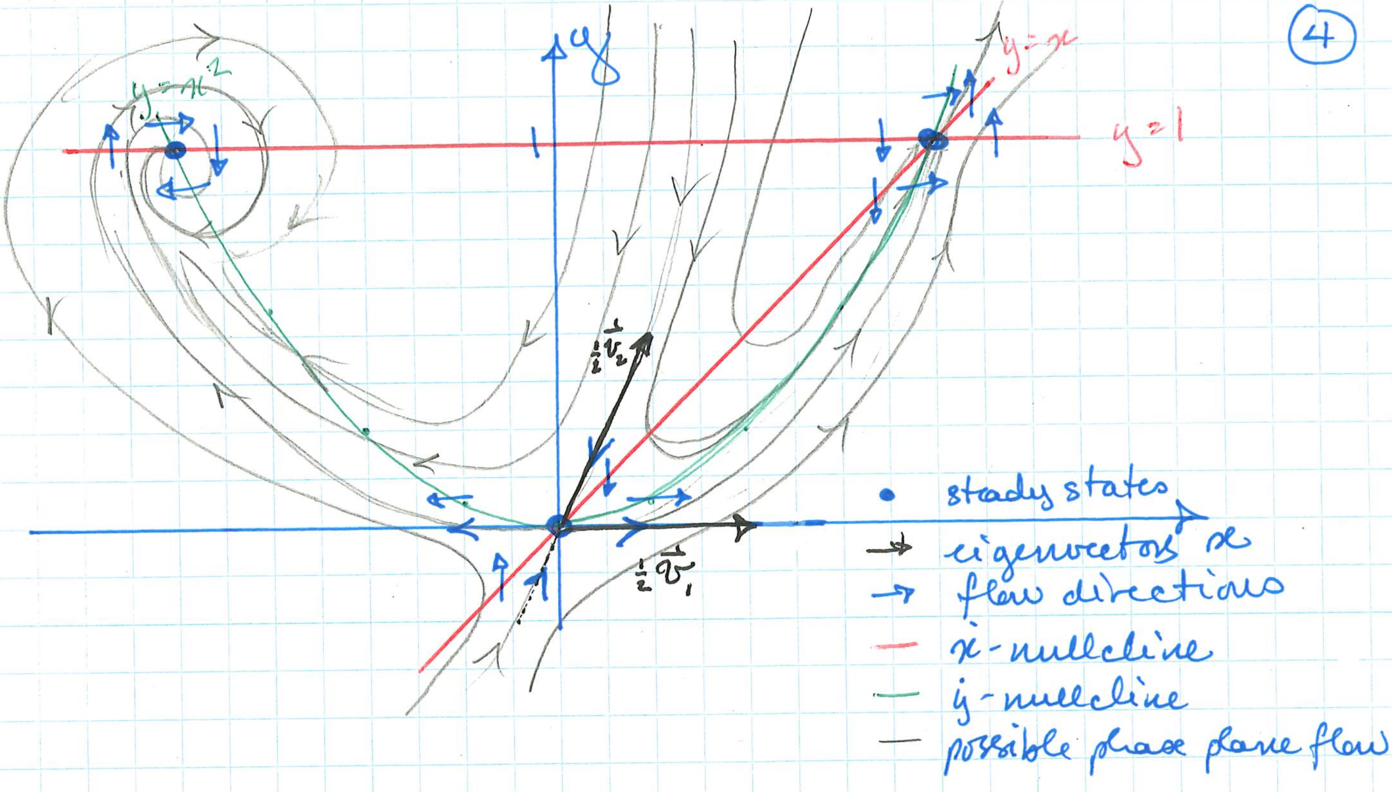
$\therefore$ the steady state is a $\boxed{\text{stable focus.}}$

At $(1,1)$

$$A = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} \quad \therefore \lambda_1 = 0, \ \lambda_2 = -1$$

The steady state is thus neutrally stable with no rotation (so, a <u>node</u>).

Next page: New sketch of the phase plane now including the stability information.



Flow across nullclines - calculations

vertical flow

Near $(-1,1)$:

On $y=1$ take $g(-1-\varepsilon,1) = (-1-\varepsilon)^2-1 = 1+2\varepsilon+\varepsilon^2-1 > 0$

$g(-1+\varepsilon,1) = (-1+\varepsilon)^2-1 = 1-2\varepsilon+\varepsilon^2-1 < 0$ (to order ε)

Near $(1,1)$

On $y=1$ take $g(1-\varepsilon,1) = (1-\varepsilon)^2-1 = 1-2\varepsilon+\varepsilon^2-1 < 0$

$g(1+\varepsilon,1) = (1+\varepsilon)^2-1 = 1+2\varepsilon+\varepsilon^2-1 > 0$

On $y=x$ try $g(1-\varepsilon,1-\varepsilon) = (1-\varepsilon)^2-(1-\varepsilon) = (1-\varepsilon)(1-\varepsilon-1)$
$= (1-\varepsilon)(-\varepsilon) < 0$

$g(1+\varepsilon,1+\varepsilon) = (1+\varepsilon)^2-(1+\varepsilon) = (1+\varepsilon)(1+\varepsilon-1)$
$= (1+\varepsilon)(\varepsilon) > 0$

We already know the behaviour on the $y=x$ nullcline near $(0,0)$ b/c of the eigenvectors.

Near $(-1,1)$

On $y=x^2$, we see that if $x<1$ and $y>1$ we have

$$f(x,y) = \underbrace{(x-y)}_{<0}\underbrace{(1-y)}_{<0} > 0$$

We expect the flow to flip at each steady state except perhaps at $(1,1)$, because two $\dot{x}=0$ nullclines intersect there.

Near (1,1)

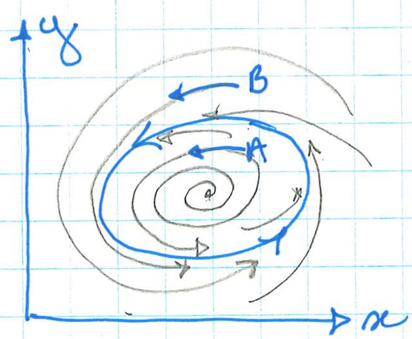
On $y=x^2$, we see that if $y>1$ and $x>1$ we have

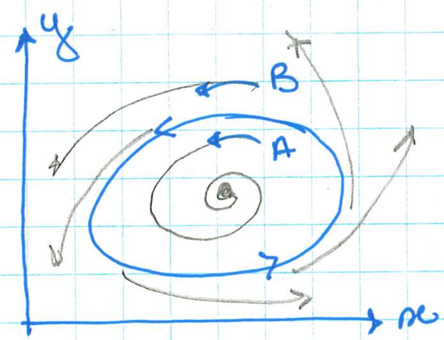
$$f(x,y) = \underbrace{(x-y)}\underbrace{(1-y)}_{<0}$$

↳ unclear, but note that $y=x^2$ lies above $y=x$, so $x<y$ and this difference must be negative $\therefore f(x,y)>0$.

So the x direction flow does not flip on the nullcline $y=x^2$ as it passes through the (1,1) steady state.

2(a)



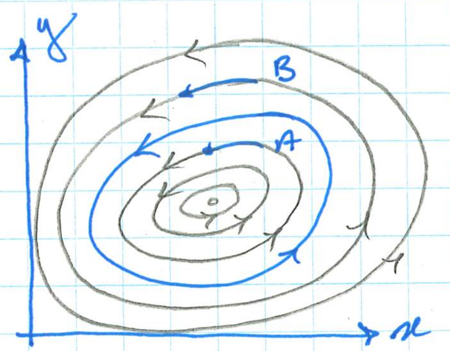


Case 1: Flow is toward the closed orbit on both sides. There is an unstable focus inside the closed orbit.

Case 2: Flow is away from the closed orbit on both sides. There is a stable focus inside the closed orbit.

Case 3: (not shown) Flow is away from the closed orbit on the outside, & toward the closed orbit on the inside. There is an unstable focus inside the closed orbit.

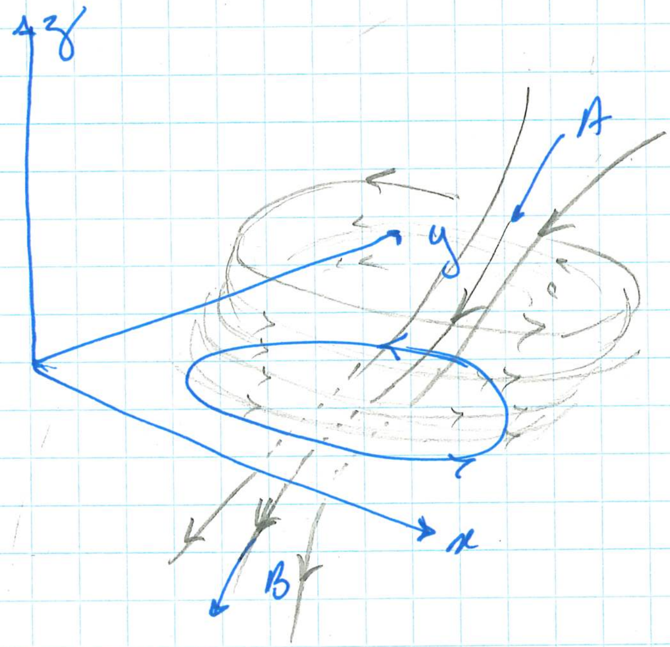
Case 4: (not shown) Flow is toward the closed orbit from the outside + away from the closed orbit on the inside. There is a stable focus inside the closed orbit.



Case 5: Concentric closed orbits. There is a centre inside the original closed orbit (indeed, inside all of the closed orbits).

(ii) We observe that all of the cases above include a steady state inside the closed orbit.

(b)



Imagine water swirling in a leaky bucket into which water is being poured from above. Some flow will go straight through & some will rotate. In a real bucket the swirling orbits wouldn't be periodic, but in a theoretical bucket they could be. Tornadoes have similar flow - there are lots of good diagrams on the web!

3. Equations (1) are

$$\begin{cases} \dfrac{dN}{dT} = rN\left(1-\dfrac{N}{K}\right) - \dfrac{cNP}{a+N} \\ \dfrac{dP}{dt} = \dfrac{bNP}{a+N} - mP \end{cases}$$

and equations (2) are

$$\begin{cases} \dfrac{dn}{dt} = n\left(1-\dfrac{n}{\gamma}\right) - \dfrac{np}{1+n} \\ \dfrac{dp}{dt} = \beta\left(\dfrac{n}{1+n} - \alpha\right)p \end{cases}$$

If we plug the scaling ratios

$$n = \frac{N}{a},\ p = \frac{c}{ra}P,\ t = rT,\ \alpha = \frac{m}{b},\ \beta = \frac{b}{r},\ \gamma = \frac{K}{a}$$

into equations (2) we obtain:

$$\begin{cases} \dfrac{d(N/a)}{d(rT)} = \dfrac{N}{a}\left(1-\dfrac{N/a}{K/a}\right) - \dfrac{N/a}{1+N/a}\dfrac{c}{ra}P \\ \dfrac{d(\frac{c}{ra}P)}{d(rT)} = \dfrac{b}{r}\left(\dfrac{N/a}{1+N/a} - \dfrac{m}{b}\right)\dfrac{c}{ra}P \end{cases} \iff (\mathrm{I})$$

$$(\mathrm{I}) \iff \begin{cases} \dfrac{1}{ar}\dfrac{dN}{dT} = \dfrac{N}{a}\left(1-\dfrac{N}{K}\right) - \dfrac{N}{a+N}\dfrac{c}{ra}P \\ \dfrac{c}{r^2a}\dfrac{dP}{dT} = \dfrac{b}{r}\left(\dfrac{N}{a+N} - \dfrac{m}{b}\right)\dfrac{c}{ra}P \end{cases}$$

$$\iff \begin{cases} \dfrac{dN}{dT} = rN\left(1-\dfrac{N}{K}\right) - \dfrac{cN}{a+N}P \\ \dfrac{dP}{dT} = \dfrac{arb}{c}\left(\dfrac{N}{a+N} - \dfrac{m}{b}\right)\dfrac{c}{ra}P \end{cases}$$

$$(1) \Leftrightarrow \begin{cases} \dfrac{dN}{dT} = rN\left(1-\dfrac{N}{K}\right) - \dfrac{cN}{a+N}P \\ \dfrac{dP}{dT} = \dfrac{bNP}{a+N} - mP \end{cases}$$

Thus, equations (2) and (1) are the same.

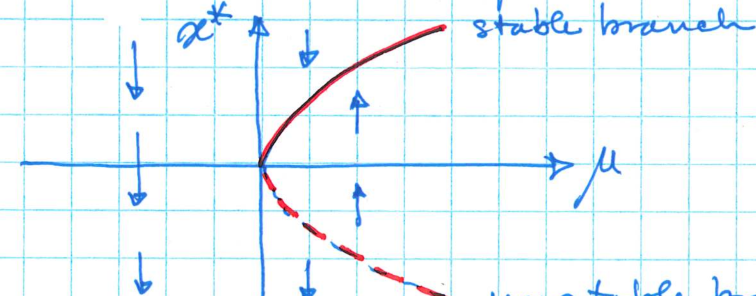
4. a) saddle-node bifurcation

$\dot{x} = \mu - x^2, \quad \mu \in \mathbb{R}$

Steady state is $\dot{x}=0 \Leftrightarrow \mu - x^2 = 0 \Leftrightarrow x^* = \pm\sqrt{\mu}$
$\therefore$ the steady state only exists if $\mu \geq 0$. Stability is determined by the sign of $\dot{x}$:

$\dot{x} > 0$ if $|x| < \sqrt{\mu}$
$\dot{x} < 0$ if $|x| > \sqrt{\mu}$

We thus arrive at the bifurcation diagram



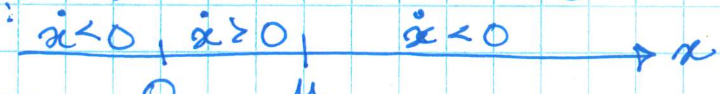
We see that the two branches collide + the steadystate disappears.

b) transcritical bifurcation

$\dot{x} = \mu x - x^2, \quad \mu \in \mathbb{R}$

Steady state is $\dot{x}=0 \Leftrightarrow \mu x - x^2 = 0 \Leftrightarrow x^* = 0$ or $x^* = \mu$.
So there are two steady states + they exist $\forall \mu \in \mathbb{R}$.
Stability is determined by the sign of $\dot{x}$:

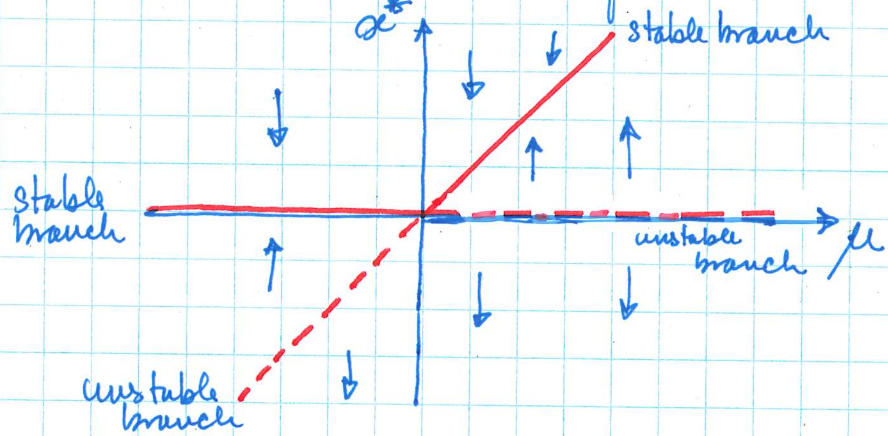
case 1:



case 2:

$\dot{x}<0$ | μ | $\dot{x}>0$ | 0 | $\dot{x}<0$ $\rightarrow x$

So we arrive at the bifurcation diagram



We see that the two branches exchange stability at $\mu=0$.

c) supercritical bifurcation

$$\dot{x} = \mu x - x^3$$

The steady states are given by

$$\dot{x}=0 \Leftrightarrow \mu x - x^3 = 0 \Leftrightarrow x(\mu - x^2) = 0$$
$$\Leftrightarrow x^* = 0 \text{ or } \pm\sqrt{\mu}$$

Stability is determined by the sign of $\dot{x}$. Note that the $\pm\sqrt{\mu}$ steady states only exist for $\mu \geq 0$.

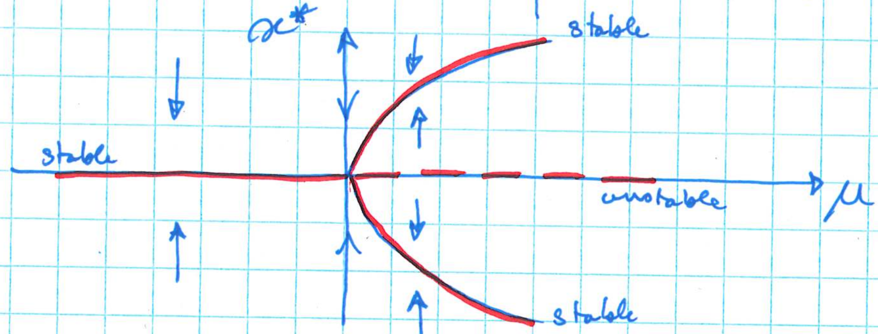
case 1: $\mu > 0$

$\dot{x}>0$ | $-\sqrt{\mu}$ | $\dot{x}<0$ | 0 | $\dot{x}>0$ | $\sqrt{\mu}$ | $\dot{x}<0$ $\rightarrow x$

case 2: $\mu < 0$

$\dot{x}>0$ | 0 | $\dot{x}<0$ $\rightarrow x$

We thus arrive at the bifurcation diagram



d) subcritical pitchfork bifurcation

$\dot{x} = \mu x + x^3$

The steady states are given by

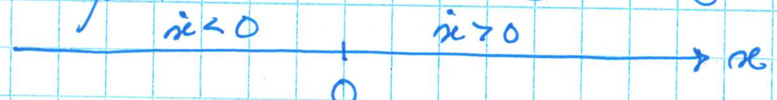
$\dot{x} = 0 \iff \mu x + x^3 = 0 \iff x(\mu + x^2) = 0$

$\iff x^* = 0$ or $x^* = \pm\sqrt{-\mu}$

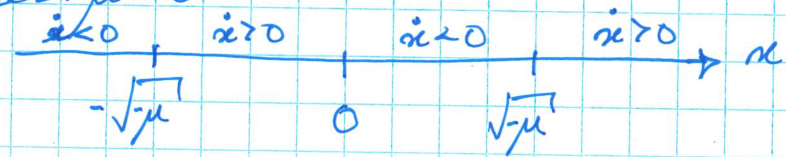
The $\pm\sqrt{-\mu}$ steady states only exist if $\mu \leq 0$.

Stability is determined by the sign of $\dot{x}$.

case 1: $\mu > 0$



case 2: $\mu < 0$



We thus arrive at the bifurcation diagram