Math 339 - Dynamical Systems
Sep-Dec 2019
Assignment # 4

Instructions: You are being evaluated on the presentation, as well as the correctness, of your answers. Try to answer questions in a clear, direct, and efficient way. Sloppy or incorrect use of technical terms will lower your mark.

1. Consider the two-species competition model

$$\dot{x} = r_1 x \left(1 - \frac{x}{\kappa_1} - \frac{\beta_{12}}{\kappa_1} y\right), \tag{1}$$

$$\dot{y} = r_2 y \left(1 - \frac{y}{\kappa_2} - \frac{\beta_{21}}{\kappa_2} x\right), \tag{2}$$

where r_i, κ_i, and β_{ij} are positive real parameters. Show that Dulac's criterion but not Bendixson's criterion can be used to establish the fact that no limit cycles exist. (*Hint: Let $B(x, y) = 1/xy$.*

2. Consider the May predator-prey model (also called the Leslie-Gower model or the May-Holling-Tanner model):

$$\dot{n} = n(1-n) - \alpha \frac{n}{\beta + n} p, \tag{3a}$$

$$\dot{p} = \rho\, p \left(1 - \frac{p}{n}\right). \tag{3b}$$

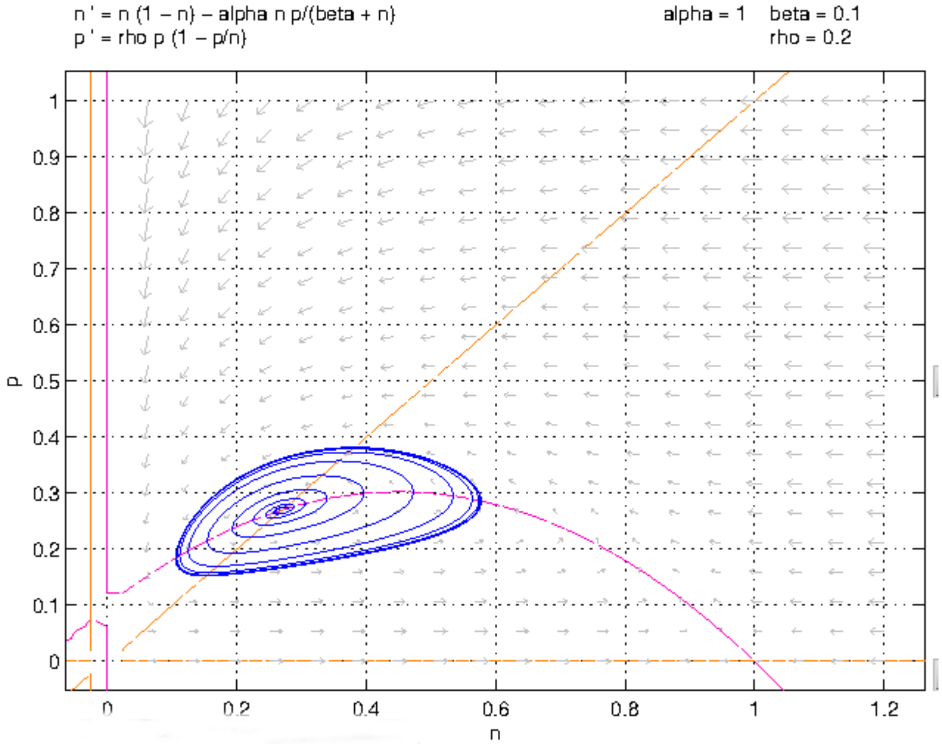
We will focus on the case where $\alpha = 1$, $\rho = 0.2$, and $\beta = 0.1$.

(a) Find the steady states and their stability (ignore the simgular point at (0,0).

(b) Show that the coexistence steady state goes through a Hopf bifurcation. (*Hint: Recall that we found the Hopf bifurcation in the previous predator-prey model by shifting the predator nullcline to the left until it fell to the left of the peak in the prey nullcline. The equivalent shift in this model is to change the angle of the predator nullcline. Thinking this way should help you select the best bifurcation parameter.*

(c) For the given parameter values, the system has the limit cycle solution shown in Figure 1. Prove the existence of the limit cycle using the Poincaré-Bendixson Theorem. Use Figure 1 to help you figure out what trapping region you should use.

Figure 1: Figure for question # 2.



3. Consider the first plankton-oxygen dynamics paper [1] (a link to the paper appears in the "Lecture Notes" web page for the course). Verify that equations (13)-(15) with equations (16)-(18) yields the dimensionless equations (19)-(21). Explain why you know that the variable c' is dimensionless.

4. Read the Introduction to [1].
   (a) What is the difference between phytoplankton and plankton?
   (b) What is the percentage of atmospheric oxygen produced by phytoplankton according to the paper (the number is higher than what I stated in class)?
   (c) What is "net oxygen production"?
   (d) Name some other effects that plankton can have on the climate (two references are given - you can find the information you need there, or through internet research at reputable sites)?

5. In [3], the authors state that with their earlier work, "it remained unclear how robust the prediction of oxygen depletion in response to a sufficiently large increase in water temperature is to the details of parametrization of the coupling between phytoplankton and oxygen." More specifically, "model prediction can only be regarded as meaningful if it does not depend strongly on the specific choice of functional feedbacks." The authors thus study seven (!) different phytoplankton-oxygen models. A summary appears in Table 1. Look at the seven different models, and explain how the functional forms were varied. (*Hint: You can think of this exercise as adding a new column to Table 1 in which the mathematical forms of the different functional responses are included.*) Plot the different functional responses using Maple (or equivalent) to show how they differ.

6. How has the study of nonlinear ODE models, and the oxygen-phytoplankton models in particular, affected your understanding of climate change models? (*One-paragraph answer (more is allowed, if you have lots to say!).*)

References

[1] Y. Sekerci and S. Petrovskii (2015) Mathematical modelling of plankton-oxygen dynamics under the climate change *Bulletin of Mathematical Biology* **77**:2325-2353.

[2] S. Petrovskii, Y. Sekerci, and E. Venturino (2017) Regime shifts and ecological catastrophes in a model of plankton-oxygen dynamics under the climate change *Journal of Theoretical Biology* **424**:91-109.

[3] Y. Sekerci and S. Petrovskii (2018) Global warming can lead to depletion of oxygen by disrupting phytoplankton photosynthesis: A mathematical modelling approach *Geosciences* **8**:201-221.