Math 339 - Dynamical Systems
Sep-Dec 2019
Assignment # 4

Instructions: You are being evaluated on the presentation, as well as the correctness, of your answers. Try to answer questions in a clear, direct, and efficient way. Sloppy or incorrect use of technical terms will lower your mark.

1. Consider the two-species competition model

$$\dot{x} = r_1 x\left(1 - \frac{x}{\kappa_1} - \frac{\beta_{12}}{\kappa_1}y\right), \tag{1}$$

$$\dot{y} = r_2 y\left(1 - \frac{y}{\kappa_2} - \frac{\beta_{21}}{\kappa_2}x\right), \tag{2}$$

where r_i, κ_i, and β_{ij} are positive real parameters. Show that Dulac's criterion but not Bendixson's criterion can be used to establish the fact that no limit cycles exist. (*Hint: Let* $B(x,y) = 1/xy$.

2. Consider the May predator-prey model (also called the Leslie-Gower model or the May-Holling-Tanner model):

$$\dot{n} = n(1-n) - \alpha\frac{n}{\beta + n}p, \tag{3a}$$

$$\dot{p} = \rho p\left(1 - \frac{p}{n}\right). \tag{3b}$$

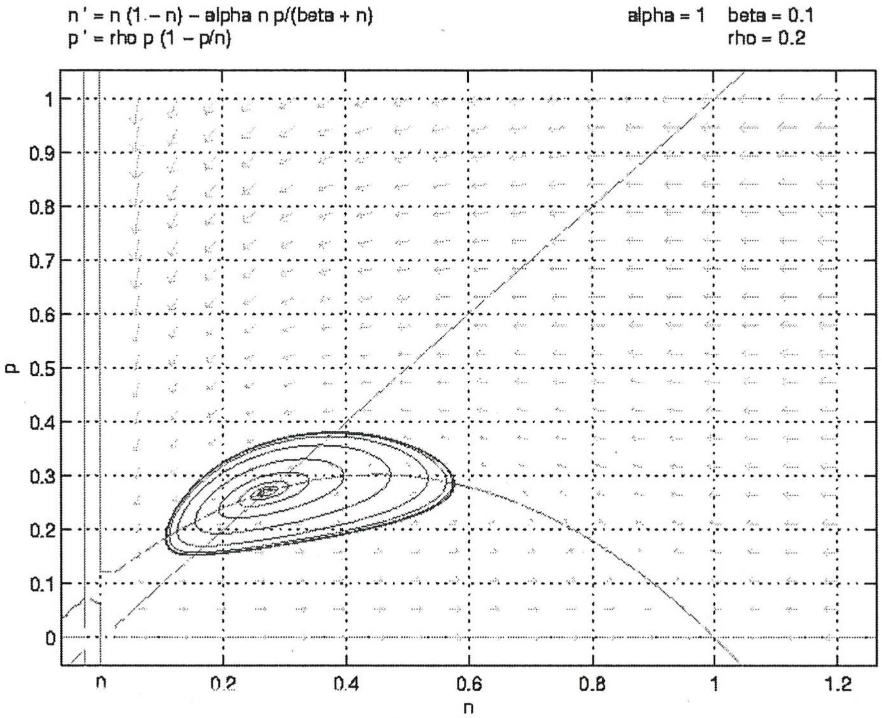
We will focus on the case where $\alpha = 1$, $\rho = 0.2$, and $\beta = 0.1$.

(a) Find the steady states and their stability (ignore the simgular point at (0,0).

(b) Show that the coexistence steady state goes through a Hopf bifurcation. (*Hint: Recall that we found the Hopf bifurcation in the previous predator-prey model by shifting the predator nullcline to the left until it fell to the left of the peak in the prey nullcline. The equivalent shift in this model is to change the angle of the predator nullcline. Thinking this way should help you select the best bifurcation parameter.*

(c) For the given parameter values, the system has the limit cycle solution shown in Figure 1. Prove the existence of the limit cycle using the Poincaré-Bendixson Theorem. Use Figure 1 to help you figure out what trapping region you should use.

Figure 1: Figure for question # 2.



3. Consider the first plankton-oxygen dynamics paper [1] (a link to the paper appears in the "Lecture Notes" web page for the course). Verify that equations (13)-(15) with equations (16)-(18) yields the dimensionless equations (19)-(21). Explain why you know that the variable c' is dimensionless.

4. Read the Introduction to [1].

   (a) What is the difference between phytoplankton and plankton?

   (b) What is the percentage of atmospheric oxygen produced by phytoplankton according to the paper (the number is higher than what I stated in class)?

   (c) What is "net oxygen production"?

   (d) Name some other effects that plankton can have on the climate (two references are given - you can find the information you need there, or through internet research at reputable sites)?

5. In [3], the authors state that with their earlier work, "it remained unclear how robust the prediction of oxygen depletion in response to a sufficiently large increase in water temperature is to the details of parametrization of the coupling between phytoplankton and oxygen." More specifically, "model prediction can only be regarded as meaningful if it does not depend strongly on the specific choice of functional feedbacks." The authors thus study seven (!) different phytoplankton-oxygen models. A summary appears in Table 1. Look at the seven different models, and explain how the functional forms were varied. (*Hint: You can think of this exercise as adding a new column to Table 1 in which the mathematical forms of the different functional responses are included.*) Plot the different functional responses using Maple (or equivalent) to show how they differ.

6. How has the study of nonlinear ODE models, and the oxygen-phytoplankton models in particular, affected your understanding of climate change models? (*One-paragraph answer (more is allowed, if you have lots to say!).*)

References

[1] Y. Sekerci and S. Petrovskii (2015) Mathematical modelling of plankton-oxygen dynamics under the climate change *Bulletin of Mathematical Biology* **77**:2325-2353.

[2] S. Petrovskii, Y. Sekerci, and E. Venturino (2017) Regime shifts and ecological catastrophes in a model of plankton-oxygen dynamics under the climate change *Journal of Theoretical Biology* **424**:91-109.

[3] Y. Sekerci and S. Petrovskii (2018) Global warming can lead to depletion of oxygen by disrupting phytoplankton photosynthesis: A mathematical modelling approach *Geosciences* **8**:201-221.

Math 339 - Fall 2019
A#4- Solutions

1. $$\begin{cases} \dot{x} = r_1 x \left(1 - \frac{x}{K_1} - \frac{\beta_{12}}{K_1} y\right) = F(x,y) \\ \dot{y} = r_2 y \left(1 - \frac{y}{K_2} - \frac{\beta_{21}}{K_2} x\right) = G(x,y) \end{cases}$$

Let D = the positive quadrant.

i) Bendixson's Criterion

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \left(r_1 - 2\frac{r_1}{K_1} x - \frac{\beta_{12}}{K_1} y\right) + \left(r_2 - 2\frac{r_2}{K_2} y - \frac{\beta_{21}}{K_2} x\right)$$

$$= \underbrace{(r_1 + r_2)}_{r} - \left(\underbrace{\left(\frac{2r_1}{K_1} + \frac{\beta_{21}}{K_2}\right)}_{a} x + \underbrace{\left(\frac{2r_2}{K_2} + \frac{\beta_{12}}{K_1}\right)}_{b} y\right)$$

This function is not identically zero except on the line

$$ax + by = r \Leftrightarrow y = -\frac{a}{b}x + \frac{r}{b}$$

so the first condition is satisfied. The second condition is not satisfied though, as the function changes sign as it crosses the line $y = -\frac{a}{b}x + \frac{r}{b}$.

ii) Dulac's Criterion

$$\frac{\partial}{\partial x}(BF) + \frac{\partial}{\partial y}(BG) = \frac{\partial}{\partial x}\left(\frac{F}{xy}\right) + \frac{\partial}{\partial y}\left(\frac{G}{xy}\right) = I$$

$$I = \frac{\partial}{\partial x} r_1\left[1 - \frac{x}{K_1} - \frac{\beta_{12}}{K_1} y\right] + \frac{\partial}{\partial y} r_2\left[1 - \frac{y}{K_2} - \frac{\beta_{21}}{K_2} x\right]$$

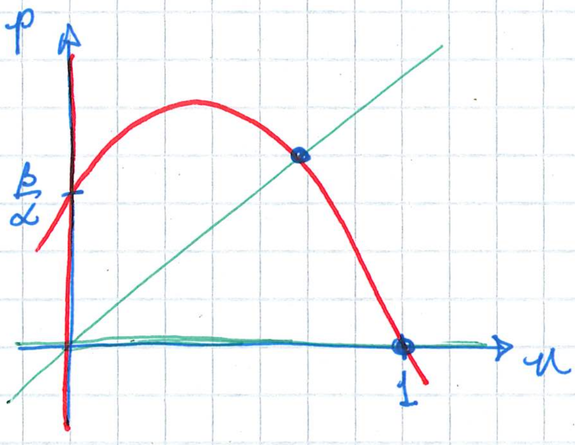
$$= -\frac{r_1}{K_1} - \frac{r_2}{K_2} < 0 \quad \forall (x,y) \in \mathbb{R}^2 \text{ or } \forall (x,y) \in D.$$

$\therefore$ Using Dulac's criterion we can rule out limit cycles.

2a) Steady States

$$\begin{cases} \dot{n} = f(n,p) \\ \dot{p} = g(n,p) \end{cases}$$

$$\begin{cases} f(n,p) = 0 \\ g(n,p) = 0 \end{cases} \Rightarrow \begin{cases} n = 0 \text{ or } (1-n)(\beta+n) = \alpha p \iff p = \dfrac{(1-n)(\beta+n)}{\alpha} \\ p = 0 \text{ or } p = n \end{cases}$$



Ignoring the ss at $(0,0)$ (which is singular in $g(n,p)$), we are left with two steady states:

$(1,0)$ and (n^*, n^*)

solving for n^*:

$$n^* = \frac{(1-n^*)(\beta+n^*)}{\alpha} \iff \text{''1''}$$

$$\begin{aligned} \text{''1''} &\iff \alpha n^* = \beta + n^* - \beta n^* - n^{*2} \\ &\iff n^{*2} + (\alpha+\beta-1)\, n^* - \beta = 0 \\ &\iff n^* = \left[(1-\alpha-\beta) \pm \sqrt{(1-\alpha-\beta)^2 + 4\beta}\right]\frac{1}{2} \quad \ldots (1) \end{aligned}$$

The coexistence steady state exists if $\beta > 0$. We take the positive root.

Stability

$$J = \begin{bmatrix} \frac{\partial f}{\partial n} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial n} & \frac{\partial g}{\partial p} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 2n - \alpha \left[\frac{\beta + n - n}{(\beta+n)^2}\right] p & \frac{-\alpha n}{\beta + n} \\ \frac{\rho p^2}{n^2} & \rho - \frac{2\rho p}{n} \end{bmatrix}$$

At $(1, 0)$

$$J = \begin{bmatrix} 1 - 2 & -\frac{\alpha}{\beta+1} \\ 0 & \rho \end{bmatrix}$$ Eigenvalues are $\lambda_1 = -1, \; \lambda_2 = \rho$

So this steady state is a <u>saddle</u>.

At (n^*, n^*)

$$J = \begin{bmatrix} 1 - 2n^* - \alpha \frac{\beta n^*}{(\beta+n^*)^2} & \frac{-\alpha n^*}{\beta + n^*} \\ \rho \frac{n^{*2}}{n^{*2}} & \rho - 2\rho \frac{n^*}{n^*} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 2n^* - \alpha\beta \frac{n^*}{(\beta+n^*)^2} & \frac{-\alpha n^*}{\beta+n^*} \\ \rho & -\rho \end{bmatrix} \quad \cdots (2)$$

Plugging in the given parameter values, $\alpha=1, \beta=0.1, \rho=2$, we have

$$u^* = \frac{1}{2}\left[(1-1-0.1)+\sqrt{(1-1-0.1)^2+4(0.1)}\right] \simeq 0.27$$

$$J \simeq \begin{bmatrix} 0.26 & -0.73 \\ 0.2 & -0.2 \end{bmatrix}$$

So the eigenvalues at $(0.27, 0.27)$ are

$$|J-\lambda I| = 0 \Leftrightarrow \lambda = \frac{1}{2}\left[Tr(J) \pm \sqrt{Tr(J)^2 - 4Det(J)}\right]$$

$$= \frac{1}{2}\left[0.06 \pm \sqrt{(0.06)^2 - 4(0.09)}\right]$$

$$= 0.03 \pm i\,0.6$$

So the coexistence steady state is an <u>unstable focus</u>.

b) Since u^* is not a function of ρ, but ρ appears in J (i.e. λ is a function of ρ), we choose ρ as our bifurcation parameter. Then we have

$$J \simeq \begin{bmatrix} 0.26 & -0.73 \\ \rho & -\rho \end{bmatrix}$$

In order for a Hopf bifurcation, we require

$$\begin{cases} Tr(J) = 0 \\ Det(J) > 0 \\ \dfrac{\partial Tr(J)}{\partial \rho} \neq 0 \end{cases} \Rightarrow \begin{cases} 0.26 - \hat{\rho} = 0 \\ \hat{\rho}(-0.26 + 0.73) \neq 0 \\ -1 \neq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \hat{\rho} = 0.26 \\ \hat{\rho} \neq 0 \\ -1 \neq 0 \end{cases}$$

All three conditions are satisfied at $\rho = \hat{\rho} = 0.26$. Thus a Hopf bifurcation occurs at $\rho = \hat{\rho} = 0.26$. It is a subcritical Hopf bifurcation ($\because \partial(Tr(J))/\partial\rho < 0$).

To show that a Hopf bifurcation occurs when p or α is used as the bifurcation parameter, we use maple. See the following two pages for the calculations.

```
> with(LinearAlgebra) :
```

First, we compute the steady state as a function of alpha and beta.

```
> nstar := ((1 - alpha - beta) + sqrt((1 - alpha - beta)^2 + 4·beta)) / 2
```

$$nstar := \frac{1}{2} - \frac{1}{2}\alpha - \frac{1}{2}\beta + \frac{1}{2}\sqrt{(1-\alpha-\beta)^2 + 4\beta} \qquad (1)$$

Then we compute the Jacobian at the steady state.

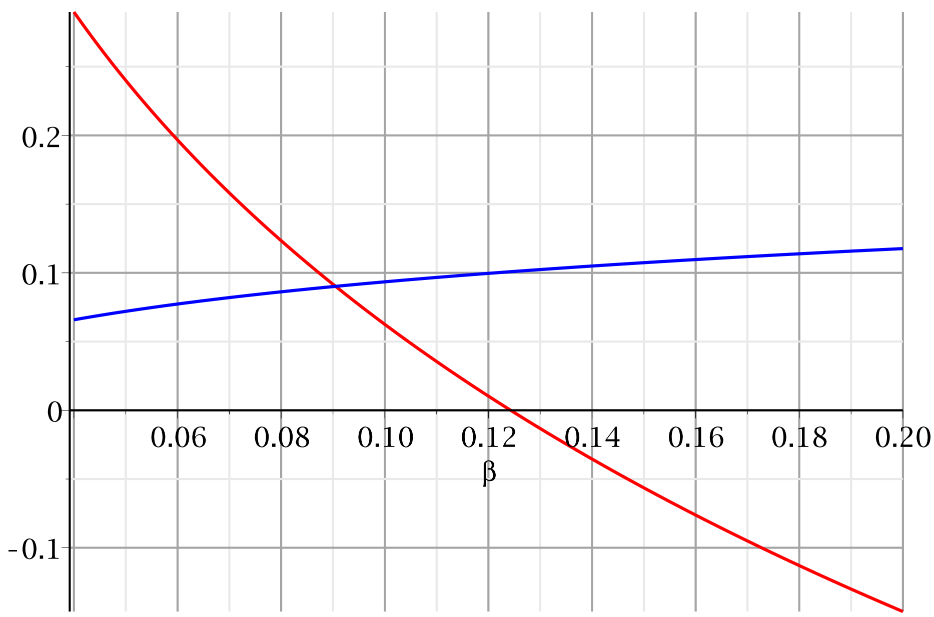
```
> J := <<1 - 2·nstar - alpha·beta·nstar/(beta + nstar)^2, rho> | <-alpha·nstar/(beta + nstar), -rho>> :
```

Then we compute the Trace and Determinant of the Jacobian, and use the unapply command to turn the Trace and Jacobian into functions of alpha, beta, and rho.

```
> TrJ := unapply(J(1, 1) + J(2, 2), alpha, beta, rho) : DetJ
      := unapply(Determinant(J), alpha, beta, rho) :
```

If we are interested in beta as a bifurcation parameter, we plot TrJ(beta) and DetJ (beta).

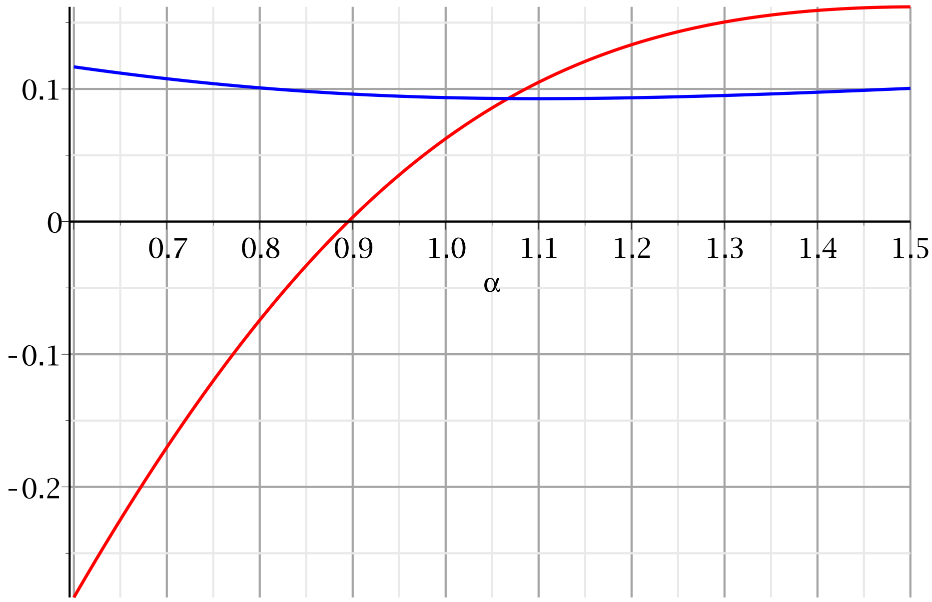
```
> plot([TrJ(1, beta, 0.2), DetJ(1, beta, 0.2)], beta = 0.04 ..0.2, colour = [red, blue],
      gridlines)
```



In order for a Hopf bifurcation to occur, we require that TrJ(beta*)=0, DetJ(beta*)>0, and the derivative of TrJ(beta) be nonzero at beta=beta*. We see that all three of these conditions are satisfied at beta*~= 0.122. Since TrJ(beta) goes from positive to negative values as beta increases, the bifurcation is a subcritical Hopf bifurcation.

If we are interested in alpha as a bifurcation parameter, then we plot TrJ(alpha) and DetJ(alpha).

> $plot([TrJ(\text{alpha}, 0.1, 0.2), DetJ(\text{alpha}, 0.1, 0.2)], \text{alpha} = 0.6..1.5, colour = [red, blue], gridlines)$



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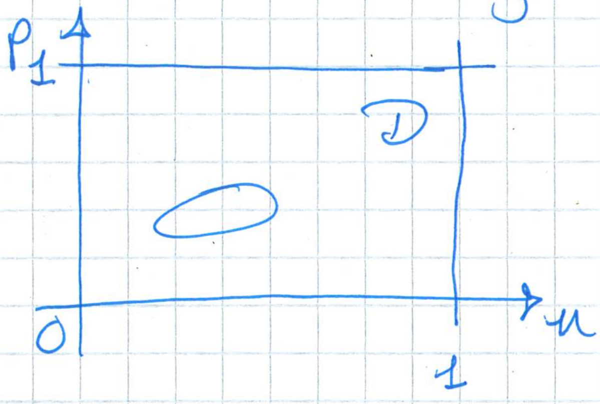
In order for a Hopf bifurcation to occur, we require that TrJ(alpha*)=0, DetJ(alpha*) >0, and the derivative of TrJ(alpha) be nonzero at alpha=alpha*. We see that all three of these conditions are satisfied at alpha*~= 0.9. Since TrJ(alpha) goes from negative to positive values as alpha increases, the bifurcation is a supercritical Hopf bifurcation.

We choose the bounding box D given by:



On $n=0$
require $\dot{n} \geq 0 \Leftrightarrow 0 \geq 0$ ✓

On $p=0$
require $\dot{p} \geq 0 \Leftrightarrow 0 \geq 0$ ✓

On $n=1$
require $\dot{n} \leq 0 \Leftrightarrow -\dfrac{1}{0.1+1}\, p \leq 0$ ✓

On $p=1$
require $\dot{p} \leq 0 \Leftrightarrow 0.2\left(1-\dfrac{1}{n}\right) \leq 0$

$\Leftrightarrow 1-\dfrac{1}{n} \leq 0$

$\Leftrightarrow 1 \leq \dfrac{1}{n}$

$\Leftrightarrow n \leq 1$ ✓

$\therefore$ Flow inside the box D is trapped, the box contains no stable steady states and one unstable steady state, & so by the Poincaré-Bendixson theorem there must be a limit cycle inside D.

3. It turns out that plugging (16)-(18) into (19)-(21) does not yield (13)-(15), and so there is a typo in the paper.

In order to determine the correct nondimensional groupings, we recompute the nondimensionalisation calculation, using (16)-(18) as a guide. We obtain

$$t' = mt, \quad c' = \frac{c}{c_0}, \quad u' = \frac{\gamma}{m} u, \quad v' = \frac{\beta\gamma}{m^2} v$$

$$\hat{c}_i = \frac{c_i}{c_0} \quad i = 1, \ldots, 4, \quad \hat{r} = \frac{r}{m}, \quad \hat{\mu} = \frac{\mu}{m}$$

$$\hat{B} = \frac{B}{m}, \quad \hat{h} = \frac{\gamma h}{m}, \quad \hat{\eta} = \frac{\beta\eta}{m}, \quad \hat{A} = \frac{A}{c_0\gamma^2}$$

$$\hat{\delta} = \frac{\delta}{c_0\gamma^2}, \quad \hat{\nu} = \frac{m\nu}{\beta\gamma^2 c_0}$$

The groupings circled in red indicate those that need to be corrected in the paper.

(My calculations are provided, FYI, at the end of this solution set.)

4. a) Phytoplankton can photosynthesize. "Plankton" includes phytoplankton & other plankton that cannot photosynthesize.

b) 70%

c) Oxygen is produced from photosynthesis, & consumed during respiration (there are other processes but these two are the main ones). Net oxygen production is oxygen produced − oxygen consumed.

5. General framework:

$$\frac{dc}{dt} = Af(c)u - M(c,u)$$
$$\frac{du}{dt} = g(c,u) - Q(c,u)$$

Model 1:
$M(c,u) = mc + uc$ $\qquad m = 1$
$Q(c,u) = \sigma u$ $\qquad \sigma = 0.1$

Model 2: $M(c,u) = mc + uc$ $\qquad m = 1$
$Q(c,u) = \dfrac{uv}{u+h}$ $\qquad h = 0.5$?
$v = 0.3$

Model 3: $M(c,u) = mc + uc$ $\qquad h = 0.5,\ \sigma = 0.1$
$Q(c,u) = \dfrac{uv}{u+h} + \sigma u$ $\qquad m = 1,\ v = 0.3$

Model 4: $M(c,u) = mc + \dfrac{uc}{c+c_2}$ $\qquad c_2 = 0.5,\ m = 1$
$Q(c,u) = \dfrac{uv}{u+h}$ $\qquad h = 0.5$?

Model 5: $M(c,u) = mc + \dfrac{uc}{c+c_2}$ $\qquad m = 1,\ c_2 = 0.5$
$Q(c,u) = \dfrac{uv}{u+h} + \sigma u$ $\qquad h = 0.5?,\ \sigma = 0.1$

Model 6: $M(c,u) = mc + \dfrac{uc}{c+c_2}$ $\qquad m = 1,\ c_2 = 0.5$
$Q(c,u) = \sigma u$ $\qquad \sigma = 0.1$

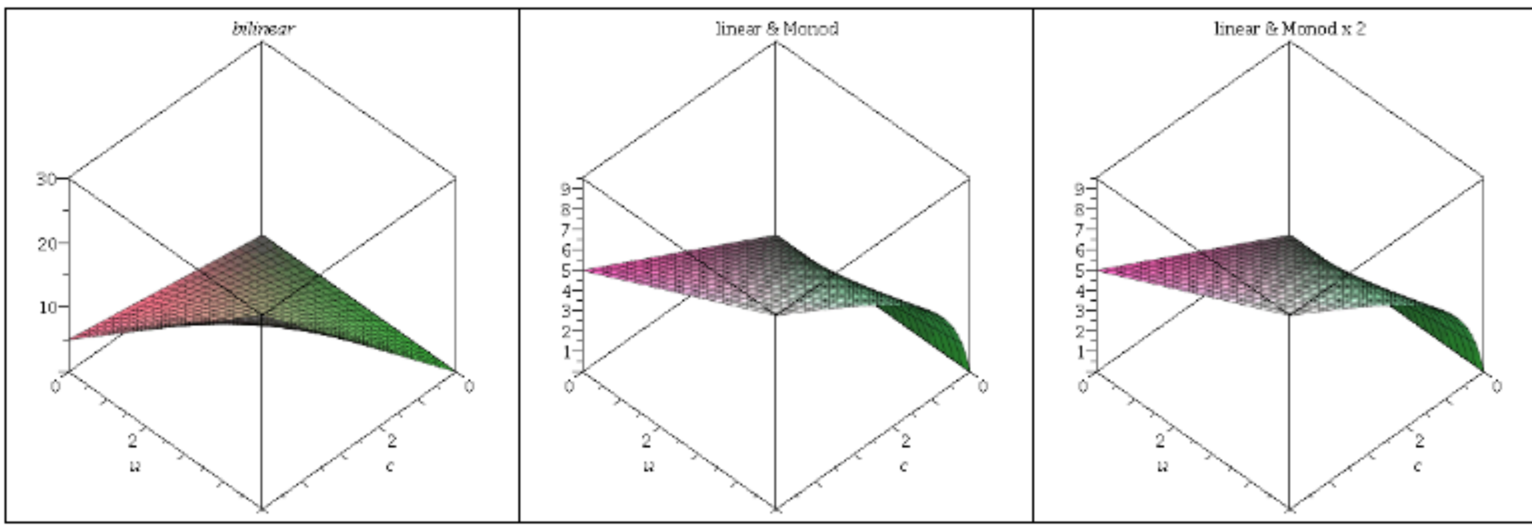
Model 7: $M(c,u) = mc + \dfrac{uc}{c+c_2} + \dfrac{\nu c v}{c+c_3}$ $\qquad m = 1,\ c_2 = 0.5,\ \nu = 0.01$
$c_3 = 1,\ v = 0.3$
$Q(c,u) = \dfrac{uv}{u+h} + \sigma u$ $\qquad h = 0.5?,\ \sigma = 0.1$

We first plot the M functions. There are three different types used in the 7 models.

> $M1 := (c, u) \rightarrow c + u \cdot c;\ M2 := (c, u) \rightarrow c + \frac{u \cdot c}{c + 0.5};\ M3 := c + \frac{u \cdot c}{c + 0.5} + \frac{0.01 \cdot 0.3 \cdot c}{c + 1};$

> $M1plot := plot3d(M1(c, u), c = 0..5, u = 0..5, axes = boxed, title = \text{'bilinear'});\ M2plot := plot3d(M2(c, u), c = 0..5, u = 0..5, axes = boxed, title = \text{"linear \& Monod"});\ M3plot := plot3d(M3(c, u), c = 0..5, u = 0..5, axes = boxed, title = \text{"linear \& Monod x 2"});$
> $Mplots := \langle \langle M1plot \rangle | \langle M2plot \rangle | \langle M3plot \rangle \rangle$

> $display(Mplots)$

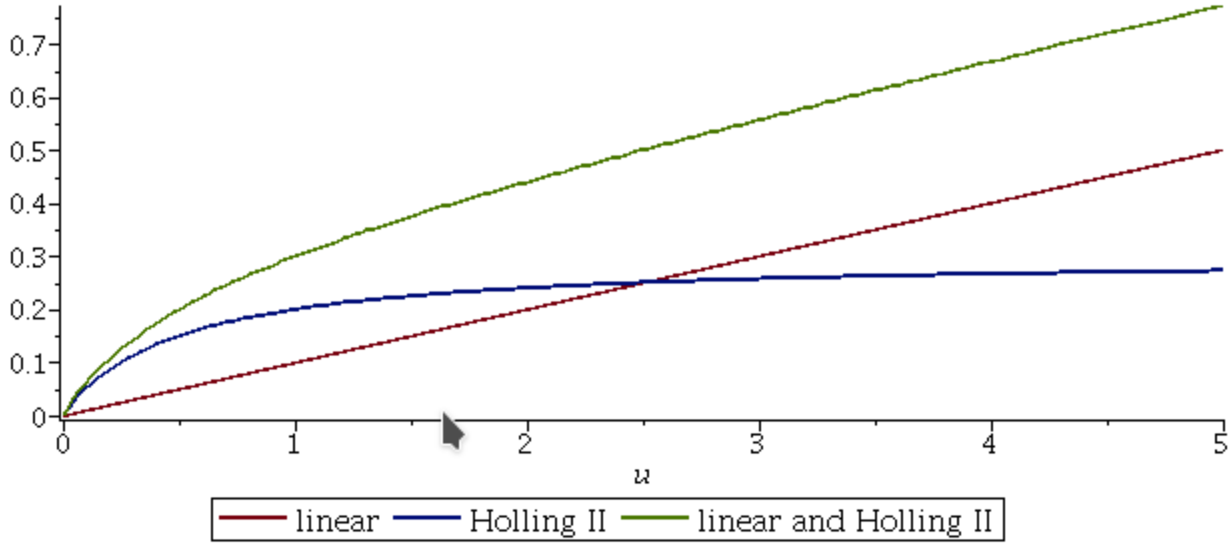


>

We now plot the Q functions. There are three different types used in the 7 different models.

> $Q1 := u \rightarrow 0.1 \cdot u;$ $Q2 := u \rightarrow \dfrac{0.3 \cdot u}{u + 0.5};$ $Q3 := u \rightarrow \dfrac{0.3 \cdot u}{u + 0.5} + 0.1 \cdot u;$

> $plot([Q1(u), Q2(u), Q3(u)], u = 0..5, axes = framed, legend = [$"linear", "Holling II", "linear and Holling II"$]);$



Finding the typo in Sekerci & Petrovskii (2015)

Model equations (13)-(15):

$$
(1)\quad
\begin{cases}
\dfrac{1}{c_0 m}\dfrac{dc}{dt} = \dfrac{1}{c_0 m}\dfrac{\frac{A c_0 u}{c_0}}{\frac{c}{c_0}+\frac{c_0}{c_0}} - \dfrac{1}{c_0 m}\dfrac{\frac{\delta u c}{c_0}}{\frac{c}{c_0}+\frac{c_2}{c_0}} - \dfrac{1}{c_0 m}\dfrac{\frac{\nu c v}{c_0}}{\frac{c}{c_0}+\frac{c_3}{c_0}} - \dfrac{mc}{m c_0} \\[3ex]
\dfrac{1}{m}\dfrac{du}{dt} = \dfrac{1}{m}\left(\dfrac{\frac{Bc}{c_0}}{\frac{c}{c_0}+\frac{c_1}{c_0}} - \gamma_u\right)u - \dfrac{1}{m}\dfrac{\beta u v}{u+h} - \dfrac{\sigma u}{m} \\[3ex]
\dfrac{1}{m}\dfrac{dv}{dt} = \dfrac{\eta\frac{c^2}{c_0^2}}{\frac{c^2}{c_0^2}+\frac{c_4^2}{c_0^2}}\,\dfrac{1}{m}\dfrac{\beta u v}{u+h} - \dfrac{\mu v}{m}
\end{cases}
$$

So far, we have

$$(2)\quad \ldots\ t' = mt,\ c' = \frac{c}{c_0},\ \hat{\sigma} = \frac{\sigma}{m},\ \hat{\mu} = \frac{\mu}{m},\ \hat{c}_i = \frac{c_i}{c_0}\ \text{for } i=1,\ldots,4$$

With these dimensionless groupings, (1) becomes

(3)
$$
\begin{cases}
\dfrac{dc'}{dt'} = \dfrac{A}{c_0 m}\,\dfrac{u}{c'+1} - \dfrac{\delta}{c_0 m}\,\dfrac{uc'}{c'+\hat{c}_2} - \dfrac{\nu}{c_0 m}\,\dfrac{c'v}{c'+\hat{c}_3} - c' \\[2ex]
\dfrac{du}{dt'} = \left(\dfrac{B}{m}\,\dfrac{c'}{c'+\hat{c}_1} - \dfrac{\gamma}{m}u\right)u - \dfrac{\beta}{m}\,\dfrac{uv}{u+h} - \hat{\sigma}u \\[2ex]
\dfrac{dv}{dt'} = \eta\,\dfrac{c'^2}{c'^2+\hat{c}_4^2}\,\dfrac{\beta}{m}\,\dfrac{uv}{u+h} - \hat{\mu}v
\end{cases}
$$

~~We know that the LHS of (3a) is dimensionless, so each term on the RHS must also be dimensionless. We ∴ set~~

~~(4) ∴~~ $\cancel{u' = \frac{\delta}{c_0 m} u \quad \text{and} \quad v' = \frac{\nu}{c_0 m} v}$

~~(5) ∴~~ $\cancel{\hat{A} = \frac{A}{c_0 m}}$

~~Plugging (4) & (5) into (3), we obtain~~

~~$\frac{dc'}{dt'} = $~~

Following Sekerci & Petrovskii (2015), we seek ways to nondimensionalise u using γ, & v using β.

Let $u' = \frac{u}{\hat{u}}$. Plugging this into (3b) we obtain

$$(4) \cdots \quad \frac{1}{\hat{u}}\frac{du}{dt'} = \left(\frac{B}{m}\frac{c'}{c'+\hat{c}_1} - \hat{u}\frac{\gamma}{m}\frac{u}{\hat{u}}\right)\frac{u}{\hat{u}} - \frac{\beta}{m}\frac{u c' v}{\hat{u}\left(\frac{u}{\hat{u}}+\frac{h}{\hat{u}}\right)\hat{u}} - \hat{\sigma}\frac{u}{\hat{u}}$$

which becomes

$$(5) \cdots \quad \frac{du'}{dt'} = \left(\frac{B}{m}\frac{c'}{c'+\hat{c}_1} - \frac{\gamma\hat{u}}{m}u'\right)u' - \frac{\beta}{m}\frac{1}{\hat{u}}\frac{u'c'v}{u'+\hat{h}} - \hat{\sigma}u'$$

where

$$\hat{h} = \frac{h}{\hat{u}}.$$

$\because$ the LHS of (5) is now dimensionless, we know that the RHS is also dimensionless. So we choose

$$
(6) \ldots \left\{
\begin{aligned}
&\hat{B} = \frac{B}{m}, \quad \hat{u} = \frac{m}{\gamma} \Rightarrow u' = \frac{\gamma u}{m} \\
&v' = \frac{B}{m\hat{u}} v = \frac{B}{m}\frac{\gamma}{m} v = \frac{B\gamma}{m^2} v \\
&\hat{h} = \frac{h\gamma}{m}
\end{aligned}
\right.
$$

Plugging (2) + (6) into (3) we obtain

$$
(7) \ldots \left\{
\begin{aligned}
&\frac{dc'}{dt'} = \frac{A}{c_0 m}\frac{u\hat{u}}{\hat{u}}\frac{1}{c'+1} - \frac{\delta}{c_0 m}\frac{c'}{c'+\hat{c}_2}\frac{u}{\hat{u}}\hat{u} - \frac{\nu}{c_0 m}\frac{1}{c'+\hat{c}_3}\frac{\beta\gamma}{m^2} v \frac{m^3}{\beta\gamma} - c' \\
&\frac{du'}{dt'} = \left(\hat{B}\frac{c'}{c'+\hat{c}_1} - u'\right)u' - \frac{u'v'}{u'+\hat{h}} - \hat{\rho}u' \\
&\frac{dv'}{dt'} = \frac{\beta}{m}\frac{(c')^2}{(c')^2+\hat{c}_4^2}\frac{u'}{u'+\hat{h}} v' - \hat{\mu} v'
\end{aligned}
\right.
$$

Finally, ∵ the LHS of (7) is dimensionless, we know that the RHS is dimensionless, which gives us

(8) $$\hat{\eta} = \frac{\eta\beta}{m}, \quad \hat{A} = \frac{A}{c_0 m}\hat{u} = \frac{A}{c_0 m}\frac{m}{\gamma} = \frac{A}{c_0\gamma}$$
$$\hat{\delta} = \frac{\delta}{c_0 m}\hat{u} = \frac{\delta}{c_0\gamma}, \quad \hat{\nu} = \frac{\nu}{c_0 m}\frac{m^2}{\beta\gamma} = \frac{\nu m}{\beta c_0\gamma}$$

Plugging (8) into (7) we obtain

(9) $$\begin{cases} \dfrac{dc'}{dt'} = \hat{A}\dfrac{u'}{c'+1} - \hat{\delta}\dfrac{c'u'}{c'+\hat{c}_2} - \dfrac{\hat{\nu}c'v'}{c'+\hat{c}_3} - c' \\[2ex] \dfrac{du'}{dt'} = \left(\dfrac{\hat{B}c'}{c'+\hat{c}_1} - u'\right)u' - \dfrac{u'v'}{u'+\hat{h}} - \hat{\sigma}u' \\[2ex] \dfrac{dv'}{dt'} = \hat{\eta}\dfrac{(c')^2}{(c')^2+\hat{c}_4^2}\dfrac{u'v'}{u'+\hat{h}} - \hat{\mu}v' \end{cases}$$

Equations (9) are the same as equations (19) - (20) in Sekerci + Petrovskii (2015).

So the correct dimensionless groupings are:

$$t' = mt, \quad c' = \frac{c}{c_0}, \quad u' = \frac{\gamma}{m} u, \quad v' = \frac{\beta\gamma}{m^2} v$$

$$\hat{c}_i = \frac{c_i}{c_0}, \; i=1,\dots,4, \quad \hat{\rho} = \frac{\rho}{m}, \quad \hat{\mu} = \frac{\mu}{m}$$

$$\hat{B} = \frac{B}{m}, \quad \hat{h} = \frac{\gamma h}{m}, \quad \hat{\eta} = \frac{\beta\gamma}{m}, \quad \hat{A} = \frac{A}{c_0 \gamma}$$

$$\hat{\delta} = \frac{\delta}{c_0 \gamma}, \quad \hat{\nu} = \frac{m\nu}{\beta\gamma c_0}$$

The groupings that differ from those given in the paper are circled in red.