Math 339 - Dynamical Systems
Sep-Dec 2019
Assignment # 5

Instructions: You are being evaluated on the presentation, as well as the correctness, of your answers. Try to answer questions in a clear, direct, and efficient way. Sloppy or incorrect use of technical terms will lower your mark.

1. Consider the second order system

$$\dot{x} = y, \qquad \dot{y} = -x - x^3.$$

   (a) Linearize the system about the (0,0) steady state and try to determine its stability using the eigenvalues of the Jacobian.

   (b) Find a Lyapunov function $\mathcal{L}(x, y)$ for the (0,0) steady state.

2. Consider the system

$$\dot{x} = -z + \left(\frac{x^3}{3} - x\right), \tag{1a}$$

$$\dot{z} = x. \tag{1b}$$

   (a) Show that $V(x, z) = x^2 + z^2$ is a Lyapunov function.

   (b) Find the largest possible open disc contained in the basin of attraction of the origin.

   (c) Use your work to discuss the stability of the solutions of the associated backwards-time problem

$$\dot{x} = z - \left(\frac{x^3}{3} - x\right), \tag{2a}$$

$$\dot{z} = -x. \tag{2b}$$

3. Find the ω-limit sets of the orbits of

$$\dot{r} = r(r-1)(3-r), \tag{3a}$$

$$\dot{\theta} = 1. \tag{3b}$$

4. Consider the predator-prey model

$$\dot{n} = n\left(1 - \frac{n}{\gamma}\right) - \frac{np}{1+n}, \tag{4a}$$

$$\dot{p} = \beta\left(\frac{n}{1+n} - \alpha\right)p, \tag{4b}$$

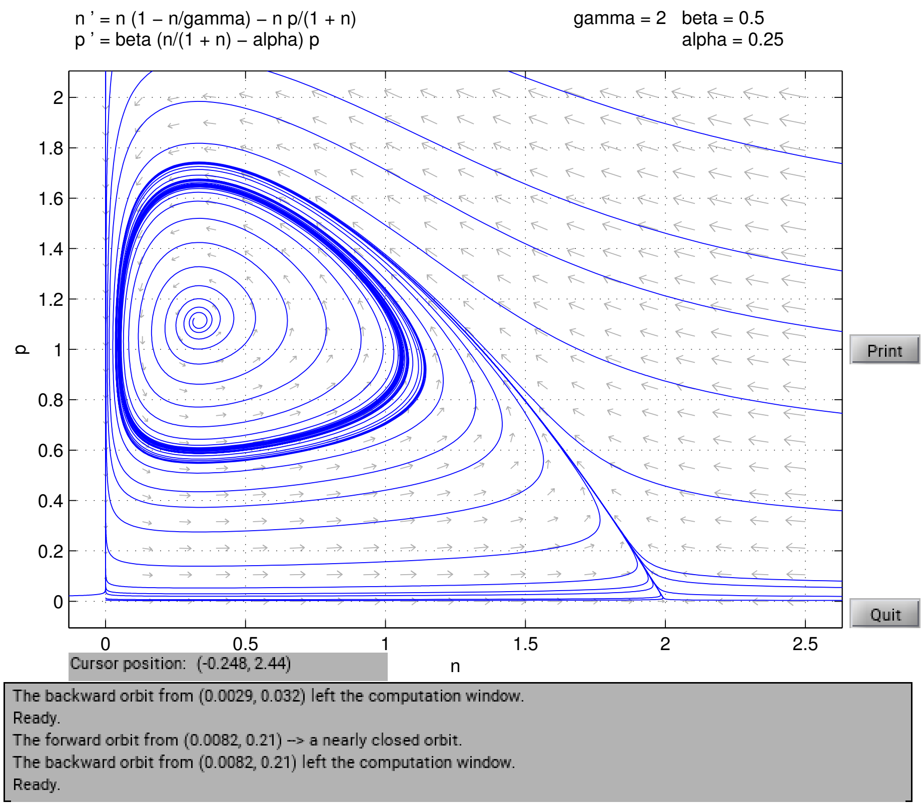
$$\tag{4c}$$

   which you saw in Assignment #3. For parameter values $\gamma = 2$, $\beta = 0.5$, and $\alpha = 0.25$, the model has three steady states and a limit cycle that is globally attracting for $n > 0$ and $p > 0$. The steady states and their linear stability are

(0,0)	saddle node,
(2,0)	saddle node,
$(\frac{1}{3}, \frac{10}{9})$	unstable focus.

The phase plane, with solution trajectories, is shown in Figure 1. Using the phase plane as a guide, give the α and ω-limit sets for all starting points in the positive quadrant. Where limit sets are difficult to define analytically, sketch them on Figure 1.

Figure 1: Figure for question # 4.



5. Consider the nonlinear system

$$\dot{x} = (x-y)(y-2), \tag{5a}$$
$$\dot{y} = x^2 - y. \tag{5b}$$

The steady states and their linear stability are

(0,0)	stable node,
(1,1)	saddle node,
$(-\sqrt{2}, 2)$	saddle node,
$(\sqrt{2}, 2)$	stable focus,

The phase plane, with solution trajectories, is shown in Figure 2. On the figure, sketch the boundaries of the various basins of attraction. Then give the ω-limit sets for points in each basin. Where necessary, sketch the limit sets on the figure.

Figure 2: Figure for question # 5.