Math 339 - Fall 2019
A#5 - Sol'ns

1. $\begin{cases} \dot{x} = y \\ \dot{y} = -x - x^3 \end{cases}$

a) $J\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1-3x^2 & 0 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Eigenvalues:

$$(-\lambda)^2 + 1 = 0 \iff \lambda^2 + 1 = 0 \iff \lambda = \pm i$$

$\therefore$ the steady state is a centre, from a linear analysis, so we don't know if, in the full model, the steady state is stable or unstable.

b) This is a Hamiltonian system, so we can use total energy as our Lyapunov function. We have

$$E = \frac{1}{2}y^2 + P(x)$$

where

$$\frac{dP}{dx} = -\ddot{x} = -\dot{y} = x + x^3 \iff (1),$$

$$(1) \iff P(x) = \frac{x^2}{2} + \frac{x^4}{4}$$

$$\therefore \quad E = \frac{1}{2}y^2 + \frac{2x^2 + x^4}{4} \quad . \quad . \quad . \quad (1)$$

Taking (1) as our Lyapunov function we have:

$$E(0,0) = 0 \quad \text{as required}$$
$$E(x,y) > 0 \quad \forall (x,y) \neq (0,0) \quad \text{as required}$$

Using $E(x,y)$ we find that

$$\dot{E}(x,y) = y\dot{y} + x\dot{x} + x^3\dot{x}$$
$$= y(-x-x^3) + xy + x^3y$$
$$= 0$$

and so the $(0,0)$ steady state is indeed a centre and solution trajectories are the level curves of $E(x,y)$.

2. $\begin{cases} \dot{x} = -z + \left(\dfrac{x^3}{3} - x\right) \\ \dot{z} = x \end{cases}$ . . . (1)

a) Consider $W(x,z) = x^2 + z^2$. We observe that the system (1) has steady states at

$\dot{z} = 0 \Leftrightarrow x = 0 \quad \therefore \quad \dot{x} = 0 \Rightarrow z = 0$

So the only steady state is at $(0,0)$.

$W(0,0) = 0$
$W(x,z) > 0 \quad \forall (x,z) \neq (0,0)$

Now consider $\dot{W}$.

$$\dot{W} = \frac{\partial W}{\partial x}\dot{x} + \frac{\partial W}{\partial z}\dot{z} = 2x\dot{x} + 2z\dot{z}$$

$$= 2x\left(-z + \left(\frac{x^3}{3} - x\right)\right) + 2zx$$

$$= \cancel{-2xz} + \frac{2x^2}{3}(x^2-3) + \cancel{2xz}$$

$$= \frac{2x^2}{3}(x^2-3)$$

$\therefore$ $W(x,z)$ is a Lyapunov function for (1) if $x^2 < 3 \Leftrightarrow |x| < \sqrt{3}$. $\because$ the level curves of $W(x,z)$ are circles, we obtain the result that $W(x,z)$ is a Lyapunov function for (1) in the disc $(x^2+z^2<3)$.

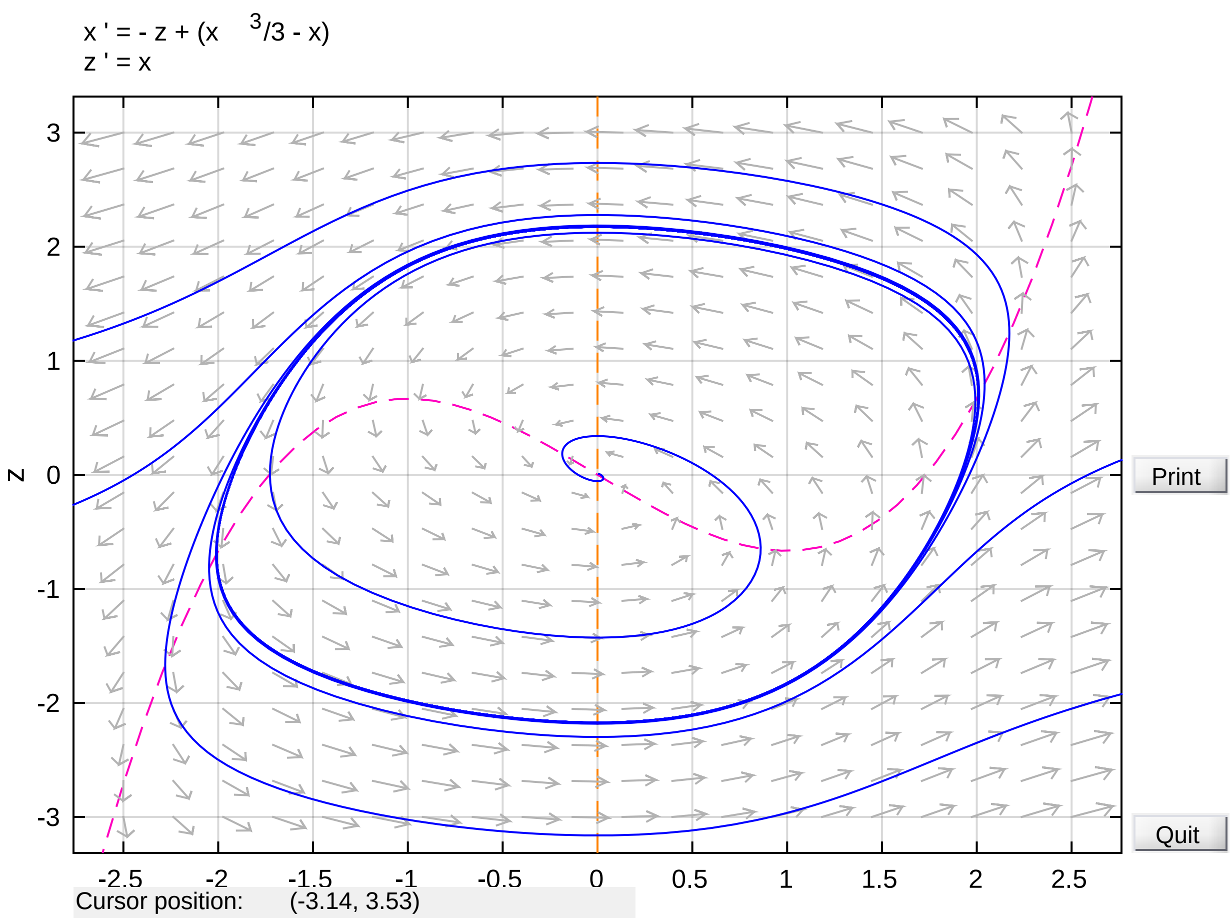
b) The Lyapunov function tells us that all starting points in the disc

$$D = \{(x,z) \in \mathbb{R}^2 \mid x^2+z^2 = 3\}$$

have ω-limit set $(0,0)$ & so the region D is contained in the basin of attraction of $(0,0)$.

c) From (b) we know that $(0,0)$ is the ω-limit set of (1) for all starting points in D. $\therefore$ $(0,0)$ is the α-limit set of the backwards system (3) for all starting points in D.

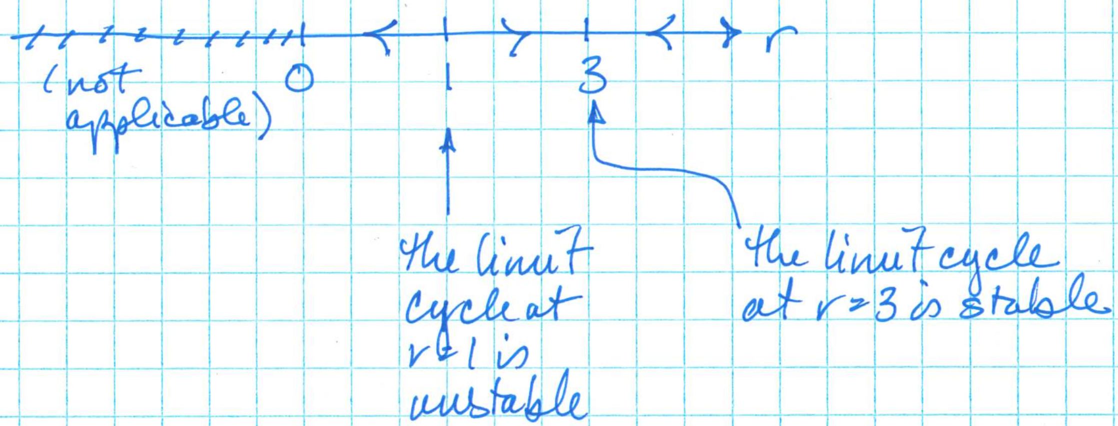
(The phase plane on the next page shows that D is indeed a subset of the basin of attraction of $(0,0)$, which is bounded by an unstable limit cycle which is not a circle & is larger than ∂D.)

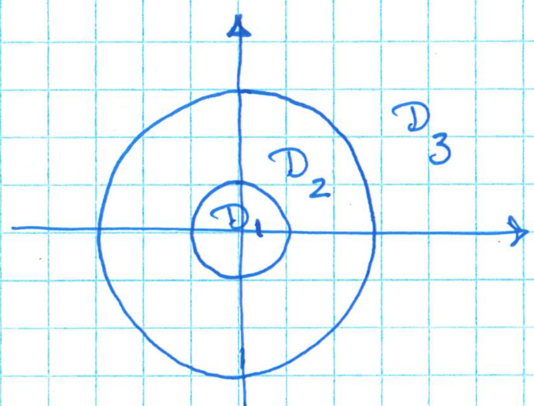


The backward orbit from (1.7, 2.3) --> a nearly closed orbit.
Ready.
The forward orbit from (-2.4, 0.061) left the computation window.
The backward orbit from (-2.4, 0.061) --> a nearly closed orbit.
Ready.

3. $\begin{cases} \dot{r} = r(r-1)(3-r) \quad \ldots\ldots \quad (3) \\ \dot{\theta} = 1 \end{cases}$

Steady state: $(0,0)$
Limitcycle: $r=1$, $r=3$, with flow:





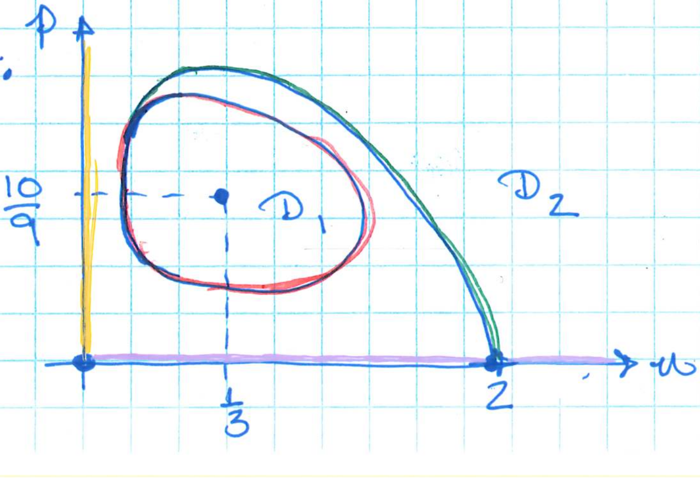
$D_1 = \{(r,\theta) \mid r<1\}$

$D_2 = \{(r,\theta) \mid 1<r<3\}$

$D_3 = \{(r,\theta) \mid r>3\}$

- $\forall$ starting points in D_1, the ω-limit set is $(0,0)$
- $\forall$ starting points in $D_2 \cup D_3 \cup \{(r,\theta) \mid r=3\}$, the ω-limit set is $r=3$, $\theta \in \mathbb{R}$.
- for $r_0 = 1$, the ω-limit set $\omega(1) = 1$

4.



= the limitcycle, D_L

= the unstable manifold of $(2,0)$, D_u

= the stable manifolds of $(2,0)$, D_S

= the stable manifold of $(0,0)$, D_E

ω-limit sets

$\forall$ starting points that are not steady states and not on D_E or D_S, the ω-limit set is D_L
" " " on D_E, the ω-limit set is $(0,0)$
" " " on D_S, " " " " " $(2,0)$

α-limit sets

$\forall$ starting points not on D_u or D_s where $n<2$, outside D_L, and not at a steady state, the α-limit set is $\emptyset$
$\forall$ starting points on D_s where $n<2$ the α-limit set is $(0,0)$
$\forall$ " " " D_u, the α-limit set is $(2,0)$
$\forall$ " " inside D_L, the α-limit set is $(\frac{1}{3}, \frac{10}{9})$

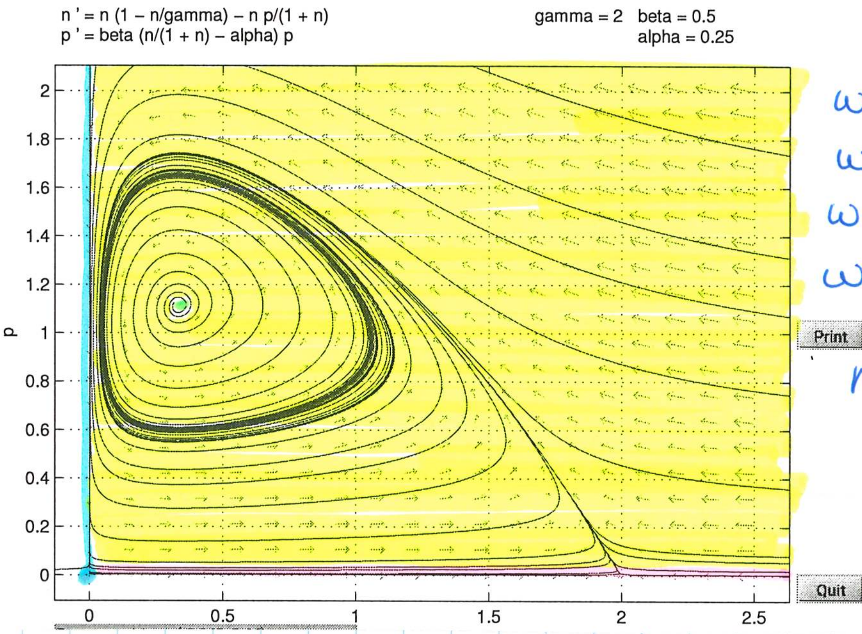
Steady States

Note that $\omega(n^*, p^*) = \alpha(n^*, p^*)$ for each of the three steady states $(n^*, p^*) = (0,0); (2,0); (\frac{1}{3}, \frac{10}{9})$.

Also, $\omega(D_L) = \alpha(D_L)$.

4. - illustrated
ω-limit sets

Figure 1: Figure for question # 4.



ω( ) = limit cycle

ω( ) = •

ω( ) = (0,0)

ω( ) = (2,0)

Note: • = (0,0)

$$\omega(n,p) = \{\text{limit cycle}\} \ \forall \{(n,p) \mid n \neq 0,\ p \neq 0 \ \& \ (n,p) \neq (\tfrac{1}{3}, \tfrac{10}{9})\}$$

$$\omega(0,p) = \{(0,0)\} \ \forall\ p \geq 0$$

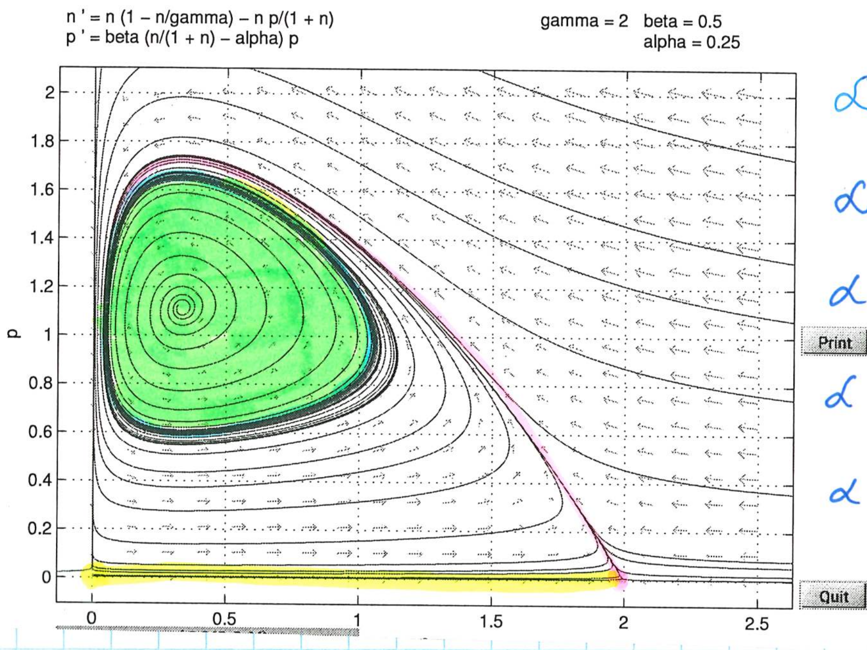
$$\omega(n,0) = \{(2,0)\} \ \forall\ n > 0$$

$$\omega\left(\tfrac{1}{3}, \tfrac{10}{9}\right) = \left(\tfrac{1}{3}, \tfrac{10}{9}\right)$$

Lea

α - limit sets

Figure 1: Figure for question # 4.



α( ) = (limit cycle)

α( ) = $(\frac{1}{3}, \frac{10}{9})$

α( ) = (2, 0)

α( ) = (0, 0)

α( ) = $\emptyset$

no highlighting

$\alpha(n,p) = \emptyset \quad \forall \{(n,p) \mid$ not on the unstable manifold of (2,0), outside the limit cycle + not on the unstable manifold of (0,0) $\}$

$\alpha(n,0) = (0,0) \quad \forall\, n < 2$

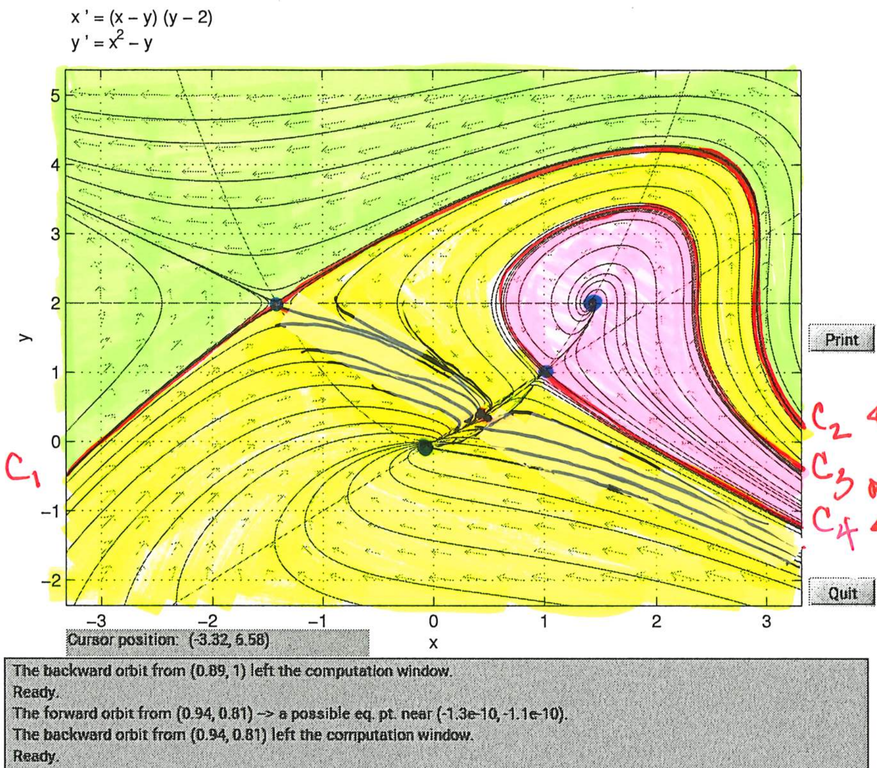
$\alpha(n,p) = (2,0) \quad \forall \{(n,p) \mid$ on the unstable manifold of (2,0) $\}$

$\alpha(n,p) =$ limit cycle $\quad \forall \{(n,p) \mid$ on the limit cycle $\}$

$\alpha(n,p) = (\frac{1}{3}, \frac{10}{9}) \quad \forall \{(n,p) \mid$ inside the limit cycle $\}$

5.

Figure 2: Figure for question # 5.



stable manifold of $(-\sqrt{2}, 2)$ → C_1

C_2 ← stable manifold of $(-\sqrt{2}, 2)$

C_3, C_4 ← stable manifolds of $(1,1)$

Note that the stable manifolds form the boundaries of the basins of attraction (regions in pink, yellow, + green highlighting).

The ω-limit set for (green) is $\emptyset$

" " " " " (yellow) is $(0,0)$

" " " " " (pink) is $(\sqrt{2}, 2)$

" " " " " C_1 is $(-\sqrt{2}, 2)$

" " " " " C_2 is $(-\sqrt{2}, 2)$

" " " " " C_3 is $(1,1)$

" " " " " C_4 is $(1,1)$

The steady state $(-\sqrt{2}, 2)$ is the (α- +) ω-limit set for $(-\sqrt{2}, 2)$.

" " " $(+\sqrt{2}, 2)$ " " (") " " " $(\sqrt{2}, 2)$.

" " " $(0,0)$ " " (") " " " $(0,0)$

" " " $(1,1)$ " " (") " " " $(1,1)$