Math 339 - Sep-Dec 2019
Lecture #13

~~② Without any restrictions on the equations, an IVP can have more than one solution.~~

~~ex: $x' = \sqrt{x}$, $x(0) = 0$~~

~~solutions are $x(t) = 0$ and $x(t) = \frac{t^2}{4}$~~

~~problem!~~

Text: Chaos, an Introduction to Dynamical Systems, Alligood, Sauer, & Yorke

3.7.4 Nonlinear Systems

③ We need to ensure continuous dependence on initial conditions (the closer two initial conditions are, the closer are their respective solutions, at least for small t).

So we need to set up a few restrictions. Consider the 1st order system

$$\begin{cases} x_1' = f_1(x_1, \dots, x_n) \\ \vdots \\ x_n' = f_n(x_1, \dots, x_n) \end{cases} \iff \vec{v}' = \vec{f}(\vec{v}) \quad \dots (1)$$

where $\vec{v} = (x_1, \dots, x_n)$ is a vector.

Thm 7.14 Existence and Uniqueness

Consider the 1^{st} order ODE (1) where both $\vec{f}$ and $\nabla \vec{f}$ are continuous on an open set $\mathcal{U}$. Then, for any real number t_0 and real vector $\vec{v}_0$, there is an open interval containing t_0 on which there exists a solution satisfying the initial condition $\vec{v}(t_0) = \vec{v}_0$, and the solution is unique.

Def 7.15

Let $\mathcal{U}$ be an open set in $\mathbb{R}^n$. A function $\vec{f}$ on $\mathbb{R}^n$ is said to be Lipschitz on $\mathcal{U}$ if there exists a constant L such that

$$|\vec{f}(\vec{v}) - \vec{f}(\vec{w})| \leq L|\vec{v} - \vec{w}|,$$

$\forall\ \vec{v}, \vec{w}$ in $\mathcal{U}$. The constant L is called a Lipschitz constant for $\vec{f}$.

doesn't grow too fast

If $\vec{f}$ has bounded 1^{st} partial derivatives in $\mathcal{U}$, then $\vec{f}$ is Lipschitz.

Ex

Consider the linear system

$$\vec{v}\,' = A\vec{v} \quad \text{where } A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}.$$

Find a Lipschitz constant for the function $A\vec{v}$ on $\mathbb{R}^2$.

Ans

$$|f(\vec{v}) - f(\vec{w})| = |A\vec{v} - A\vec{w}| = |A(\vec{v} - \vec{w})|$$

See next pages...

$$A\vec{v} = \begin{bmatrix} v_1 - 3v_2 \\ 2v_2 \end{bmatrix} \qquad A\vec{w} = \begin{bmatrix} w_1 - 3w_2 \\ 2w_2 \end{bmatrix}$$

$$|A\vec{v} - A\vec{w}| = \sqrt{(v_1 - 3v_2 - w_1 + 3w_2)^2 + (2v_2 - 2w_2)^2} \quad \ldots (1)$$

$$|\vec{v} - \vec{w}| = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2} \quad \ldots (2)$$

Require L s.t.

$$|A\vec{v} - A\vec{w}| \leq L |\vec{v} - \vec{w}| \Leftrightarrow |A\vec{v} - A\vec{w}|^2 \leq L^2 |\vec{v} - \vec{w}|^2 \quad \ldots (3)$$

We seek to rewrite the LHS of (3) in terms of $|\vec{v} - \vec{w}|^2$.

$$|A\vec{v} - A\vec{w}|^2 = ((v_1 - w_1) - 3(v_2 - w_2))^2 + 4(v_2 - w_2)^2$$

$$= (v_1 - w_1)^2 - 6(v_1 - w_1)(v_2 - w_2) + 9(v_2 - w_2)^2 + 4(v_2 - w_2)^2$$

$$= (v_1 - w_1)^2 - 6(v_1 - w_1)(v_2 - w_2) + 11(v_2 - w_2)^2$$

$$\leq 11|\vec{v} - \vec{w}|^2 + 6\max((v_1 - w_1)^2, (v_2 - w_2)^2)$$

$$\leq 11|\vec{v} - \vec{w}|^2 + 6|\vec{v} - \vec{w}|^2$$

$$\leq 17|\vec{v} - \vec{w}|^2$$

So if we choose $L = \sqrt{17}$ we know that Def 7.15 is satisfied.

$\therefore$ the linear system is Lipschitz

§ 7.5 - Motion in a Potential Field

Recall: Exact equations: We solve the ODE by recasting its solutions as the level curve of a 2-D fn $F(x,y)$.

Here: Recast the ODEs as ~~motion~~ describing movement in a potential field.

We will develop these ideas using a key example: The pendulum equation

$$\ddot{x} + k\sin(x) = 0 \quad \ldots \quad (1)$$

$k = \frac{g}{l}$

For simplicity: $k=1$.

Normal Form:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin(x) \end{cases} \quad \ldots \quad (2)$$

Suppose we wanted to find the steady states & stability:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ \sin(x) = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = \pm n\pi \end{cases}, \ n \in \mathbb{N}$$

$$J = \begin{bmatrix} 0 & 1 \\ -\cos(x) & 0 \end{bmatrix}$$

$$|J - \lambda I| = 0 \iff \begin{vmatrix} -\lambda & 1 \\ -\cos(x) & -\lambda \end{vmatrix} = 0 \iff \lambda^2 + \cos(x) = 0$$

$$\iff \lambda^2 = -\cos(x)$$

At $x = \pm 2m\pi$ (even values of n), we have

$\cos(x) = \cos(\pm 2m\pi) = 1 \therefore \lambda = \pm i$

At $x = \pm(2m+1)\pi$ (odd values of n), we have

$\cos(x) = \cos(\pm(2m+1)\pi) = -1 \therefore \lambda = \pm 1$

Thus, the fixed pts $(\pm(2m+1)\pi, 0)$ are saddles

" " " $(\pm 2m\pi, 0)$ are ?!?

Current tools: insufficient.

So we look for a different function to work with.

Idea: In non-dissipative systems, total energy is constant. What is the total energy for this system?

$$E(x,y) = \frac{1}{2}y^2 + 1 - \cos(x) \quad \cdots\cdots \quad (3)$$

Check that E is conserved ($\dot{E} = 0$):

$$\begin{aligned} \frac{dE}{dt} &= y\dot{y} + \sin(x)\,\dot{x} \\ &= y(-\sin(x)) + \sin(x)\,y \\ &= 0 \end{aligned}$$

$\therefore$ the curves $E(x,y) = c$ are level curves of E, and the solutions of the ODE system (2) must follow the level curves of E.

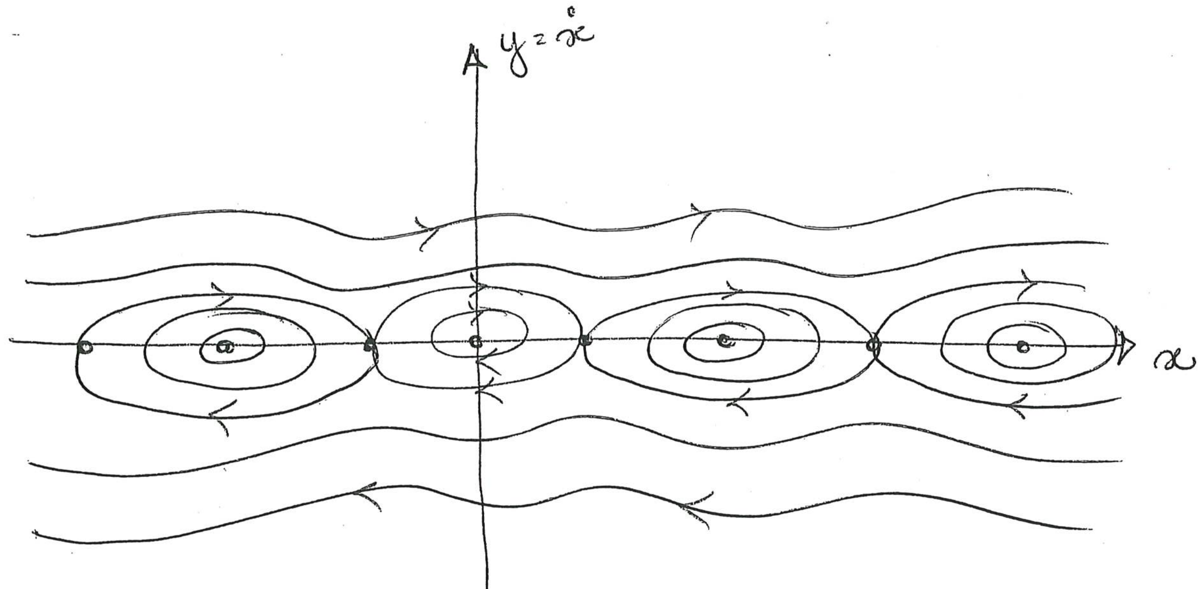
Using E

Notice $\min(E(x,y)) = 0$. These points are the fixed points of (2).

What do these curves look like?

$$E = c \iff \frac{1}{2}y^2 + 1 - \cos(x) = c$$

$$\iff y^2 = 2\,(c - 1 + \cos(x))$$



The plot of the E = c curves (above) becomes the phase plane of (2) when arrows are added.

Example of a Hamiltonian System.

General Case: The equation

$$\ddot{x} + \frac{\partial P}{\partial x} = 0 \quad \text{(4)}$$

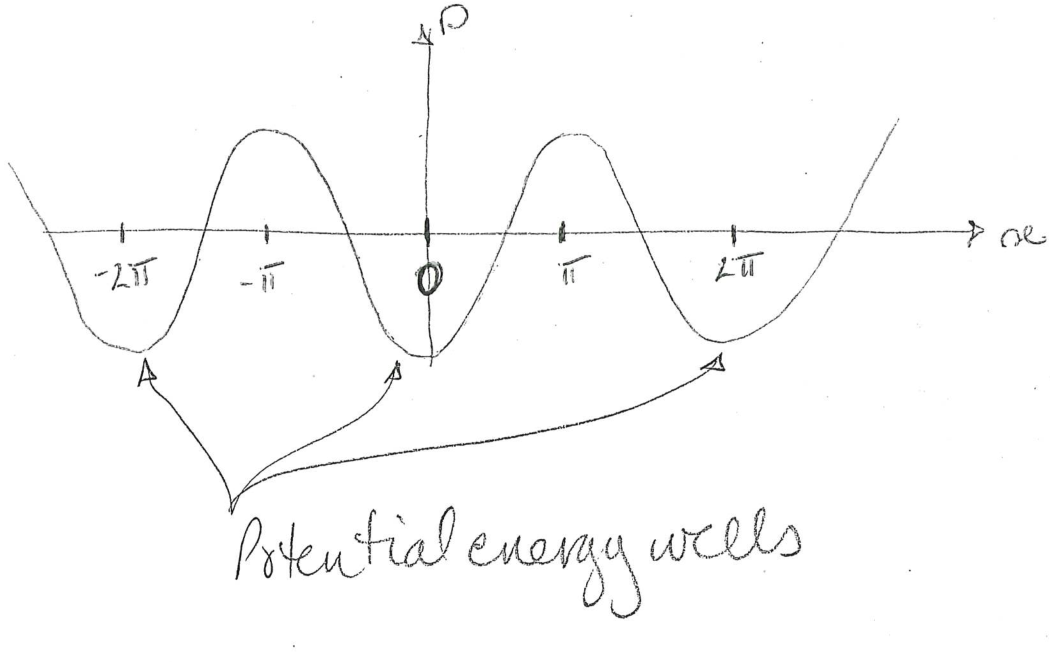
governs motion in a potential field.

$F = ma \Rightarrow a \propto F = \underbrace{\frac{\partial P}{\partial x}}_{\text{gradient of the potential field}}$

Ex1: P for the pendulum:

$\ddot{x} + k\sin(x) = 0 \Leftrightarrow \ddot{x} + k\frac{\partial}{\partial x}\left(-\cos(x)\right) = 0$

$\therefore P(x) = -\cos(x)$



P for general case: (constant energy curves)

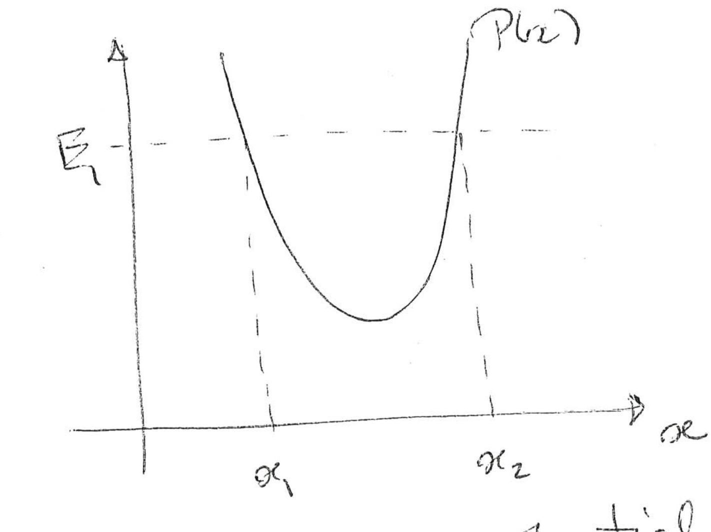
$$\dot{x}\left(\ddot{x} + \frac{\partial P}{\partial x}\right) = 0 \Leftrightarrow \ddot{x}\,\dot{x} + \frac{\partial P}{\partial x}\dot{x} = 0$$

$$\Leftrightarrow \frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{\partial P}{\partial x}\frac{dx}{dt} = 0$$

$$\Leftrightarrow \int \frac{d}{dt}\left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right) + \int \frac{d}{dt}\left(P(x)\right) = \int 0$$

$$\Leftrightarrow \underbrace{\frac{1}{2}(\dot{x})^2}_{\text{kinetic energy}} + \underbrace{P(x)}_{\text{potential energy}} = \underbrace{E_1}_{\text{ttl energy}} \quad \ldots\ldots (5)$$

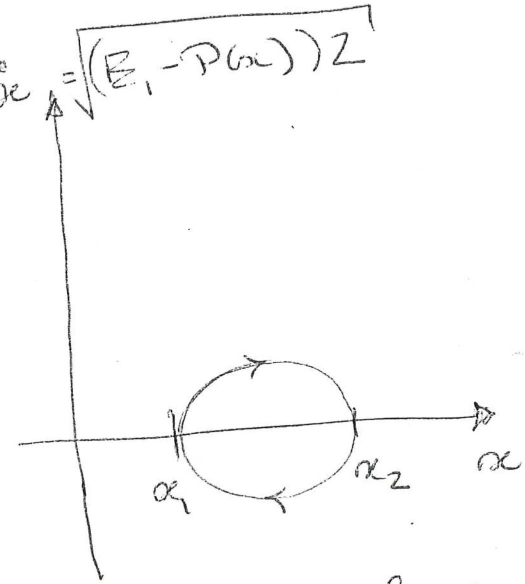
Typical:



single-well potential function

$\Rightarrow$

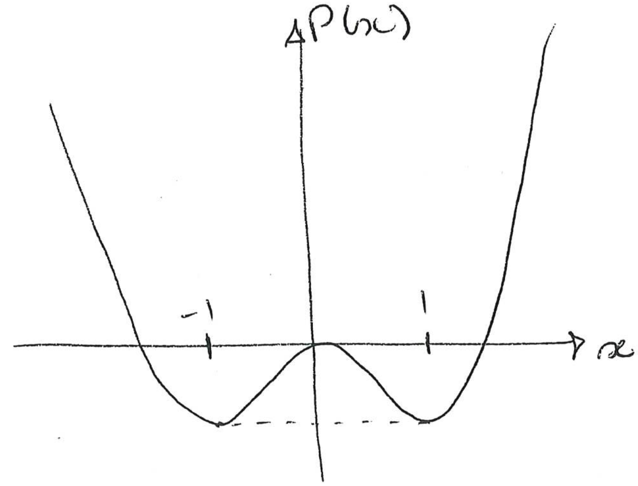
$$\dot{x} = \sqrt{(E_1 - P(x))2}$$



phase plane solution

Ex 2

$$P(x) = \frac{x^4}{4} - \frac{x^2}{2}$$



double-well potential

The motion in this potential field is given by (4):

$$\ddot{x} + \frac{\partial P}{\partial x} = 0 \iff \ddot{x} + x^3 - x = 0 \quad \ldots (6)$$

double-well Duffing eqn

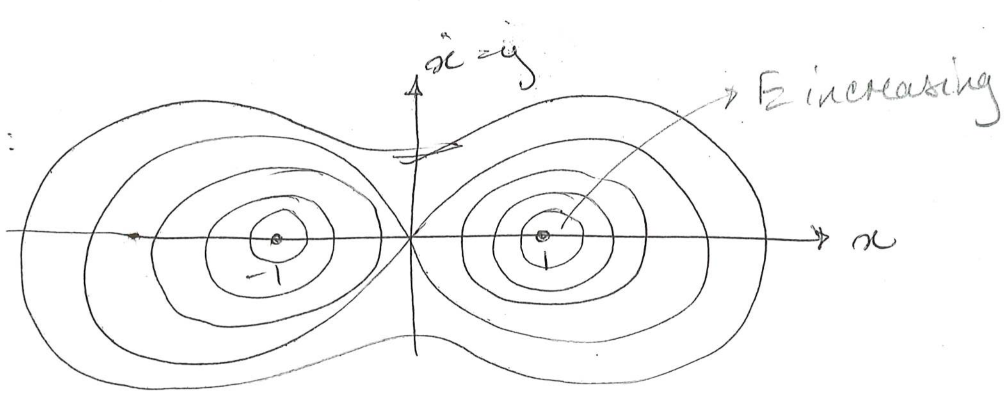
and level curves of E given by (5)

$$\frac{1}{2}(\dot{x})^2 + \frac{x^4}{4} - \frac{x^2}{2} = E_1$$

For this model,

If $\begin{cases} E_1 < 0, & \text{then soln is trapped in a well.} \\ E_2 > 0, & \text{then " circles ~~through~~ around both wells} \end{cases}$

phase plane:



Ex 3

If damping (dissipation) is added, then energy is no longer conserved.

Duffing eqn w/ damping

$$\ddot{x} + c\dot{x} + \frac{\partial P}{\partial x} = 0$$

and then

$$\dot{E} = \dot{x}\ddot{x} + \frac{\partial P}{\partial x}\dot{x}$$

$$= \dot{x}\left(-c\dot{x} - \frac{\partial P}{\partial x}\right) + \frac{\partial P}{\partial x}\dot{x}$$

$$= -c(\dot{x})^2$$

$\therefore$ total energy decreases along orbits.

phase plane: