# §7.6 Lyapunov Functions

Lyapunov functions: Alexander Mikhailovich Lyapunov (Russian)

The Lyapunov function is a generalisation of the energy function we discussed last time.

Energy function:
- is minimum (has minima) at the stable ~~a~~ steady states
- if there is dissipation, $\dot{E}<0$ in the basin of attraction of the steady state
- is constant along phase plane trajectories if there is no dissipation

~~Thm 7.23~~

~~Let $\vec{x}^*$ be an equilibrium of $\dot{\vec{x}}=\vec{f}(\vec{x})$. If there exists a Lyapunov function for $\vec{x}^*$, then $\vec{x}^*$ is stable. If~~

## Def 7.22

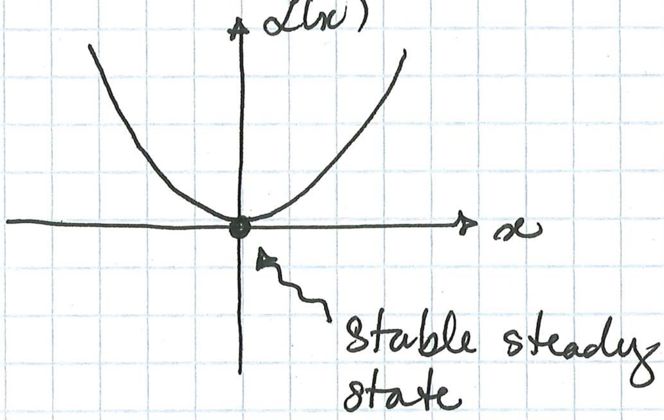
Let $\vec{x}^*$ be an equilibrium of $\dot{\vec{x}}=\vec{f}(\vec{x})$. A function $\mathcal{L}:\mathbb{R}^n \to \mathbb{R}$ is called a Lyapunov function for $\vec{x}^*$ if for some neighbourhood $W$ of $\vec{x}^*$, the following conditions are satisfied:

1. $\mathcal{L}(\vec{x}^*)=0$, and $\mathcal{L}(\vec{x}^*)>0 \ \forall \vec{x} \neq \vec{x}^*$ in $W$, &
2. $\dot{\mathcal{L}}(\vec{x}) \leq 0 \ \forall \vec{x}$ in $W$.

If the stronger inequality

2'. $\dot{\mathcal{L}}(\vec{x}) < 0 \ \forall \vec{x}$ in $W$

holds, then $\mathcal{L}$ is called a strict Lyapunov fn.




**Ex 2: pendulum from last time**

Show that for each even integer $n$, the total energy

$$E(x,y) = \frac{1}{2}y^2 + 1 - \cos(x)$$

is a Lyapunov function for the equilibria $(n\pi, 0)$ of

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin(x). \end{cases}$$

Ans

1. $E(n\pi, 0) = \frac{1}{2}(0)^2 + 1 - \cos(n\pi) = 0 + 1 - 1 = 0$ ✓

   $\uparrow$ even

   $E(x,y) > 0 \ \forall \ x \neq n\pi$ ✓

2. $\dot{E}(x,y) = 0 \ \forall \ (x,y) \in \mathbb{R}^2$

$\therefore$ $E(x,y)$ is a Lyapunov function for each steady state $(n\pi, 0)$, $n$ even.

## Theorem 7.23

Let $\vec{x}^*$ be an equilibrium of $\dot{\vec{x}} = \vec{f}(\vec{x})$. If there exists a Lyapunov function for $\vec{x}^*$, then $\vec{x}^*$ is stable. If there exists a strict Lyapunov function for $\vec{x}^*$, then $\vec{x}^*$ is asymptotically stable.

(pf: Hirsch & Smale, 1974)

## Ex 3

Show that $E(x) = x^2$ is a strict Lyapunov fn for the equilibrium $x=0$ in $\dot{x} = -x^3$.

First, consider a linear analysis

$$\dot{x} = f(x) = -x^3 \qquad \text{linearised:} \quad \dot{x} = \underbrace{f'(x^*)}_{=0} x = -3x^{*2} \cdot x$$

So the linear analysis is inconclusive. Now we use Lyapunov function analysis. Using $E(x) = x^2$ we check conditions 1 + 2 of Def 7.22:

1. $E(0) = 0$ ✓ $\quad E(x) > 0 \ \forall x \neq 0$ ✓
2. $\dot{E} = 2x\dot{x} = 2x(-x^3) = -2x^4 < 0 \ \forall x \neq 0$

$\therefore$ $E(x)$ is a strict Lyapunov function for $x^* = 0$ & by Thm 7.23 we know that $x^* = 0$ is asymptotically stable (a.s.).

## Ex 4

Let

$$\begin{cases} \dot{x} = -x^3 + xy, \\ \dot{y} = -y^3 - x^2. \end{cases} \qquad \cdots\cdots (1)$$

Prove that $(0,0)$ is an a.s.e. of (1).

Try $\mathcal{L}(x,y) = x^2 + y^2$.

1. $\mathcal{L}(0,0) = 0$, $\mathcal{L}(x,y) > 0 \ \forall (x,y) \neq (0,0)$

2. $\dot{\mathcal{L}}(x,y) = 2x\dot{x} + 2y\dot{y}$

$$= 2x(-x^3 + xy) + 2y(-y^3 - x^2)$$

$$= -2x^4 + \cancel{2x^2y} - 2y^4 - \cancel{2x^2y}$$

$$= -2(x^4 + y^4) < 0 \ \forall (x,y) \neq (0,0)$$

$\therefore \mathcal{L}(x,y) = x^2 + y^2$ is a strict Lyapunov fn for $(0,0)$ and the equilibrium is a.s.

**Def 7.24**

Let $\vec{x}^*$ be an a.s.e. of $\dot{\vec{x}} = \vec{f}(\vec{x})$. Then the <u>basin of attraction</u> of $\vec{x}^*$, denoted $\mathcal{B}(\vec{x}^*)$, is the set of initial conditions $\vec{x}_0$ such that

$$\lim_{t \to \infty} \vec{F}(t, \vec{x}_0) = \vec{x}^*.$$

Note: Any set $W$ on which $E$ is a strict Lyapunov function for $\vec{x}^*$ will be a <u>subset</u> of $\mathcal{B}(\vec{x}^*)$.

**Def 7.25**

A set $U \subset \mathbb{R}^n$ is called a <u>forward invariant set</u> for $\dot{\vec{x}} = \vec{f}(\vec{x})$ if for each $\vec{x}_0 \in U$, the forward orbit $\{\vec{F}(t, \vec{x}_0) : t \geq 0\}$ is contained in $U$. A forward invariant set that is bounded is called a <u>trapping region</u>. We also require that a trapping region be an $n$-dimensional set.

Trapping regions: Weaker notion of containment than $\mathcal{B}(\vec{x})$.