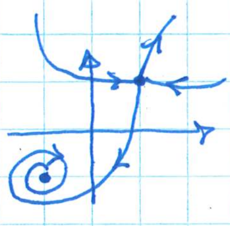
§8.1 Limit Sets for Planar ODEs



$\dot{\vec{v}} = \vec{f}(\vec{v})$ ← no "t", so autonomous . . . (1)

↑ differentiable or @ least Lipschitz (to guarantee ~~[illegible]~~ uniqueness)

Notation:
For a point $\vec{v}_0$ in $\mathbb{R}^n$, we write $\vec{F}(t,\vec{v}_0)$ for the unique solution of (1) satisfying $\vec{F}(0,\vec{v}_0) = \vec{v}_0$.

Def 8.1

A point $\vec{z}$ in $\mathbb{R}^n$ is in the <u>ω-limit set $\omega(\vec{v}_0)$</u> of the solution curve $\vec{F}(t,\vec{v}_0)$ if there is a sequence of points increasingly far out along the orbit (i.e., as $t \to +\infty$) which converges to $\vec{z}$.

Specifically, $\vec{z}$ is in $\omega(\vec{v}_0)$ if there exists an unbounded increasing sequence $\{t_n\}$ of real numbers with $\lim\limits_{t\to+\infty} \vec{F}(t_n,\vec{v}_0) = \vec{z}$.

A point $\vec{z}$ is in the <u>α-limit set $\alpha(\vec{v}_0)$</u> if there exists an unbounded <u>decreasing sequence</u> $\{t_n\}$ of real numbers ($t_n \to -\infty$) with $\lim\limits_{t\to-\infty} \vec{F}(t_n,\vec{v}_0) = \vec{z}$.

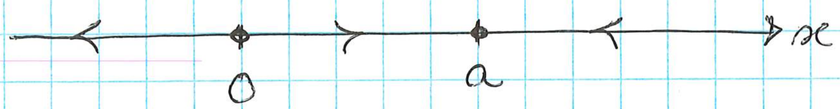
ω-limit set = forward limit set
α- " " = backward " "

Implications:
- If $\vec{v}_0$ is a steady state, then $\omega(\vec{v}_0) = \alpha(\vec{v}_0) = \{\vec{v}_0\}$.
- For <u>any</u> $\vec{v}_0$, the α-limit set of $\dot{\vec{v}} = \vec{f}(\vec{v})$ is the ω- " " " $\dot{\vec{v}} = -\vec{f}(\vec{v})$.

Example 8.2

$$\dot{x} = x(a-x), \quad a>0$$

Steady states: $x^* = 0$ or $x^* = a$

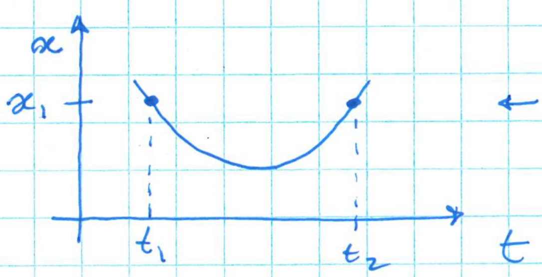


$\therefore$ $\omega(x_0) = \{a\} \ \forall x_0 > 0$
$\omega(0) = 0$ (also $\alpha(0) = 0$)
$\omega(x_0) = \{\emptyset\} \ \forall x_0 < 0$

Similarly
$\alpha(x_0) = \{\emptyset\} \ \forall x_0 > a$
$\alpha(a) = \{a\}$ (also $\omega(a) = a$)
$\alpha(x_0) = \{0\} \ \forall x_0 < a$

Thm 8.3 All solutions of the scalar differential equation $\dot{x} = f(x)$ are either monotonic increasing or monotonic decreasing as a function of t. For $x_0 \in \mathbb{R}$, if the orbit $\vec{F}(t, x_0)$, $t \geq 0$, is bounded, then $\omega(x_0)$ consists solely of an equilibrium.

pf



← Plot shows $f(x_1) < 0$ @ t_1
$f(x_1) > 0$ @ t_2

But $f(x_1)$ is unique!

$\therefore$ the fn shown cannot be the solution of $\dot{x} = f(x)$.

In sum: Solutions of scalar ODEs either $\to \infty$ or $\to$ an eqm pt. For 2-D systems, a third behaviour is possible: Periodic solutions.

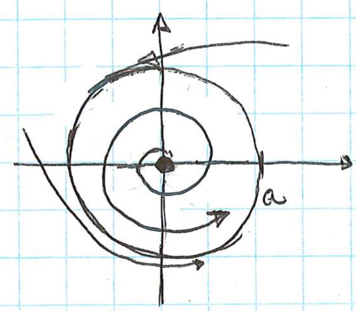
Def 8.4

If there exists a $T>0$ such that $\vec{F}(t+T,\vec{v}_0) = \vec{F}(t,\vec{v}_0)$ for all t, and if $\vec{v}_0$ is not an equilibrium, then the solution $\vec{F}(t,\vec{v}_0)$ is called a periodic orbit or cycle. The smallest number T is called the period of the orbit.

By uniqueness of solutions, periodic orbits are all simple (i.e., they do not cross).

Example 8.5

$$\begin{cases} \dot{r} = r(a-r) & a>0 \\ \dot{\theta} = b & b>0 \end{cases}$$



Steady-states: $r=0$
limit cycle: $r=a$

What are the ω-limit sets?

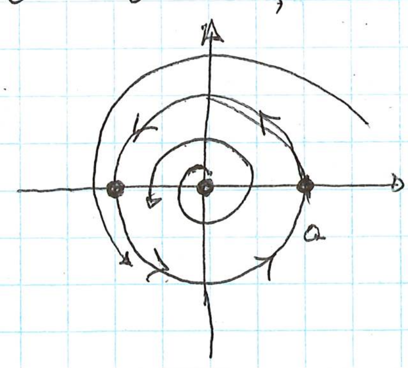
$\omega(0) = \{0\}$
$\omega(r_0,\theta_0) = \{r=a\} \quad r_0 \neq 0$

Example 8.6

$$\begin{cases} \dot{r} = r(a-r) & a>0 \\ \dot{\theta} = \sin^2(\theta) + (r-a)^2 \end{cases}$$

Steady-states: $r=0$
$r=a,\ \sin^2(\theta)=0$ for $\theta = n\pi,\ n\in\mathbb{Z}$

What are the ω-limit sets?



$\omega(0) = \{0\}$
$\omega(r=a, 0<\theta<\pi) = \{-a\}$
$\omega(r=a, \pi<\theta<2\pi) = \{a\}$

$\omega(a, 2n\pi) = \{a\}, \ n \in \mathbb{Z}$
$\omega(-a, (2n+1)\pi) = \{-a\}, \ n \in \mathbb{Z}$
$\omega(r_0, \theta_0) = \{a\}$ or $\{-a\}, \ r_0 \neq 0, a$

(book says $\omega(r_0, \theta_0) = \{r = a\} \ \forall \ r_0 \neq 0$ or a)

Properties of ω-limit sets

1. Existence: The ω-limit set of a bounded orbit is non-empty.
2. Closure: An ω-limit set is closed.
3. Invariance: If $\vec{y}$ is in $\omega(\vec{v}_0)$, then the entire orbit $\vec{F}(t, \vec{v}_0)$ is in $\omega(\vec{v}_0)$.
4. Connectedness: The ω-limit set of a bounded orbit is connected.
5. Transitivity: If $\vec{z}$ is in $\omega(\vec{y})$ and $\vec{y}$ is in $\omega(\vec{v}_0)$, then $\vec{z}$ is in $\omega(\vec{v}_0)$.

Now consider the behavior of Lyapunov functions on limit sets.

- The value of $\mathcal{L}$ is constant on limit sets.
- Limit sets are invariant.

Example 5.4.4 (An Introduction to Dynamical Systems by Robinson, Pearson/Prentice-Hall))

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - 2x^3 + y(x^2 - x^4 - y^2) \end{cases} \quad \cdots \quad (1)$$

Steady States:

$\dot{x} = 0 \Leftrightarrow y = 0$

$\dot{y} = 0 \Leftrightarrow x - 2x^3 + y(x^2 - x^4 - y^2) = 0$

$\because y = 0$ we have $x - 2x^3 = 0 \Leftrightarrow x(1 - 2x^2) = 0$

$\Leftrightarrow x = 0$ or $x^2 = \frac{1}{2}$

$\therefore$ the steady states are

$(0,0)$; $(\frac{1}{\sqrt{2}}, 0)$; $(-\frac{1}{\sqrt{2}}, 0)$

Stability

$$J = \begin{bmatrix} 0 & 1 \\ 1 - 6x^2 + y(2x - 4x^3) & x^2 - x^4 - 3y^2 \end{bmatrix}$$

At $(0,0)$:

$|J - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \lambda = \pm 1$

$\therefore (0,0)$ is a saddle.

At $\left(\frac{1}{\sqrt{2}}, 0\right)$:

$$|J - \lambda I| = 0 \Leftrightarrow \left| \begin{bmatrix} -\lambda & 1 \\ 1-3 & \frac{1}{2} - \frac{1}{4} - \lambda \end{bmatrix} \right| = 0$$

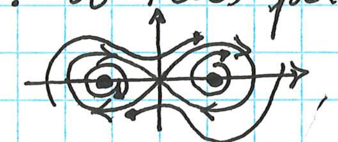
$$\Leftrightarrow \lambda^2 - \frac{1}{4}\lambda + 2 = 0$$

$$\Leftrightarrow 4\lambda^2 - \lambda + 8 = 0$$

$$\Leftrightarrow \lambda = \frac{1 \pm \sqrt{1 - 128}}{8} = \frac{1}{8} \pm \frac{\sqrt{127}}{8} i$$

$\therefore \left(0, \frac{1}{\sqrt{2}}\right)$ is an unstable focus

At $\left(-\frac{1}{\sqrt{2}}, 0\right)$:
We obtain exactly the same result, $\because J(x,0)$ only depends on x^2 and x^4. So this point is also an unstable focus.



To determine ω- & α-limit sets, observe that the system (1) would be Hamiltonian if (1b) did not depend on y. In this case, we would have

$$E = \frac{1}{2} y^2 + P(x) \quad . \quad . \quad . \quad (2)$$

where $P(x)$ is found from

$$\frac{\partial P}{\partial x} = -\ddot{x} = -\dot{y} = -x + 2x^3 \Leftrightarrow /_1$$

$$/_1 \Leftrightarrow P(x) = -\frac{x^2}{2} + \frac{x^4}{2} \quad . \quad . \quad . \quad (3)$$

Plugging (3) into (2) we obtain

$$E(x,y) = \frac{1}{2} y^2 - \frac{x^2}{2} + \frac{x^4}{2} \quad . \quad . \quad . \quad . \quad (4)$$

Suppose we use $E(x,y)$ to see how the solutions to (1) behave. Observe that

$$\begin{aligned}
\dot{E} &= y\dot{y} - x\dot{x} + 2x^3\dot{x} \\
&= y\left(x - 2x^3 + y(x^2 - x^4 - y^2)\right) - xy + 2x^3 y \\
&= y^2(x^2 - x^4 - y^2) \\
&= -2y^2 E \begin{cases} \geq 0 & \text{when } E < 0 \\ = 0 & \text{when } E = 0 \\ \leq 0 & \text{when } E < 0 \end{cases} \qquad \cdots (5)
\end{aligned}$$

(So E is a test function & not a Lyapunov function.)

Note that

$$\begin{cases} E(0,0) = 0 \\ E(\tfrac{1}{\sqrt{2}}, 0) = P(\tfrac{1}{\sqrt{2}}) = -\frac{1}{4} + \frac{1}{2}\frac{1}{4} = -\frac{1}{4} + \frac{1}{8} = -\frac{1}{8} \\ E(-\tfrac{1}{\sqrt{2}}, 0) = P(-\tfrac{1}{\sqrt{2}}) = -\frac{1}{8} \end{cases} \qquad \cdots (6)$$

and $P(x) = \frac{x^2}{2}(x^2 - 1)$ has minima at $x^2 = \frac{1}{2}$ or $x = \pm\frac{1}{\sqrt{2}}$, i.e., at the nonzero steady states. So $\min(E) = -\frac{1}{8}$.

Combining (6) with (5) we observe:

- If we start on a trajectory that includes (0,0), we remain on that trajectory b/c $\dot{E} = 0$. So the $E(0,0) = 0$ level curve is an invariant set for (1). We can solve for y in $E = 0$, & see that this level curve is bounded and one-dimensional. ∴ all trajectories must converge to the steady state at $(0,0)$ as $t \to \pm\infty$. Thus $\omega(x_0,y_0) = \alpha(x_0,y_0) = 0$ and

$$(x_0,y_0) \in W^s(0) \cap W^u(0) \quad \ldots\ldots \quad (7)$$

- If we start on a trajectory that includes $(\pm 1/\sqrt{2}, 0)$, then $E(t=0) < 0$ & so $\dot{E} \geq 0$, so must increase toward $E=0$. $\therefore$ $(\pm 1/\sqrt{2}, 0)$ is the α-limit point for all (x_0,y_0) inside the $E=0$ level curve. The ω-limit point for those same (x_0,y_0) is the origin
- If we start on a trajectory with $E(x_0,y_0) > 0$, then $\dot{E} < 0$ and E decreases to $E=0$. Thus, the ω-limit point for all starting points outside the $E=0$ level curve is the origin.