Review of the Hopf bifurcation

Ex 1: $\begin{cases} \dot{x} = \mu x - \omega y + kx(x^2+y^2) = f(x,y) \\ \dot{y} = \omega x + \mu y + ky(x^2+y^2) = g(x,y) \end{cases}$ . . . (1)

Show that this model exhibits a Hopf bifurcation. Assume $(x,y) \in \mathbb{R}^2$, $\omega > 0$

Sol'n

i) Steady States

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \iff \begin{cases} \mu x - \omega y + kx(x^2+y^2) = 0 \\ \omega x + \mu y + ky(x^2+y^2) = 0 \end{cases} \iff 1/$$

Clearly $(0,0)$ is a steady state, and if one variable is 0, the other one must be zero also. So we continue with 1/ setting $x \neq 0$ & $y \neq 0$ to see if there are any other steady states. We begin by multiplying 1/(a) by y & 1/(b) by x.

$$1/ \iff \begin{cases} \mu xy - \omega y^2 + kxy(x^2+y^2) = 0 \\ \omega x^2 + \mu xy + kxy(x^2+y^2) = 0 \end{cases}$$

$$\iff \begin{cases} \omega x^2 + \omega y^2 = 0 \\ \mu x - \omega y + kx(x^2+y^2) = 0 \end{cases}$$

$$\iff \begin{cases} x^2 + y^2 = 0 \\ \mu x - \omega y + kx(x^2+y^2) = 0 \end{cases}$$

The first equation can only be satisfied for real (x,y) if $(x,y) = (0,0)$. So there are no other steady states.

(ii) Stability of the (0,0) steady state

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \mu + 3Kx^2 + Ky^2 & -\omega + 2Kxy \\ \omega + 2Kxy & \mu + 3Ky^2 + Kx^2 \end{bmatrix}$$

$$\therefore J\Big|_{(0,0)} = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}$$

The eigenvalues of this matrix are given by

$$(\mu - \lambda)^2 + \omega^2 = 0 \iff \lambda^2 - 2\mu\lambda + \mu^2 + \omega^2 = 0 \iff$$

$$\iff \lambda^2 - 2\mu\lambda + (\mu^2 + \omega^2) = 0$$

$$\iff \lambda = \mu \pm \sqrt{\mu^2 - (\mu^2 + \omega^2)}$$

$$= \mu \pm \sqrt{-\omega^2}$$

$$= \mu \pm i\omega$$

We see that the eigenvalues are pure imaginary when $\mu = 0$, so there is the potential for a Hopf bifurcation.

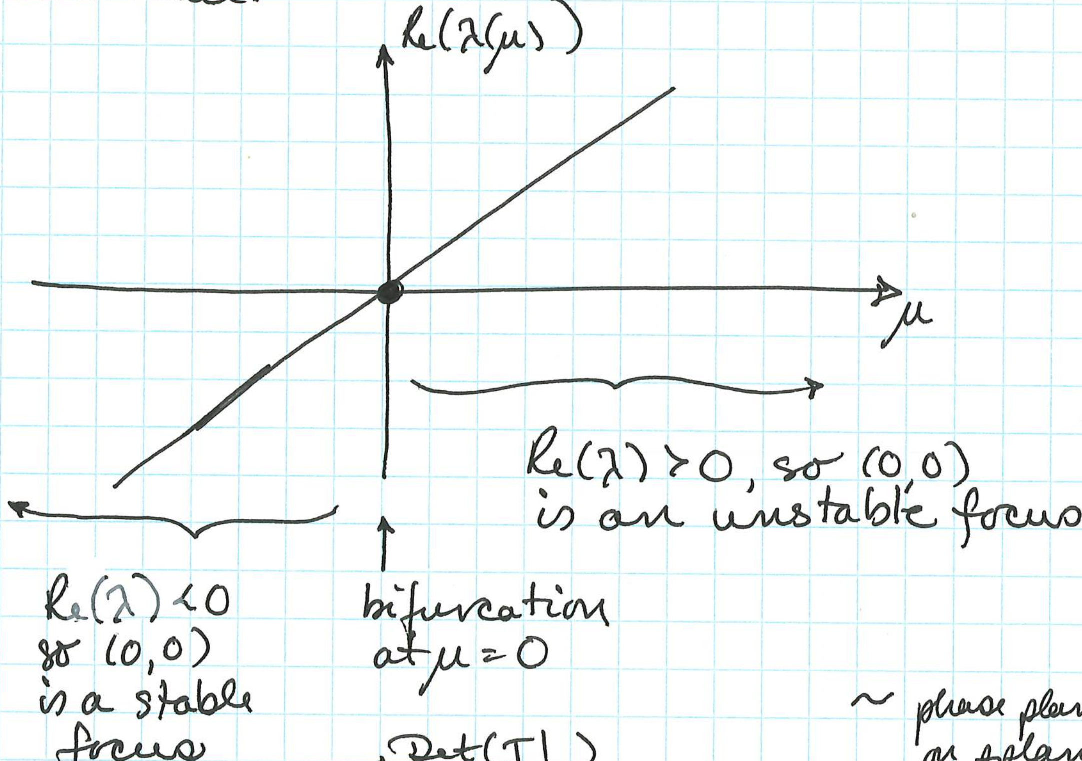
iii) Hopf bifurcation at $\hat{\mu}=0$?

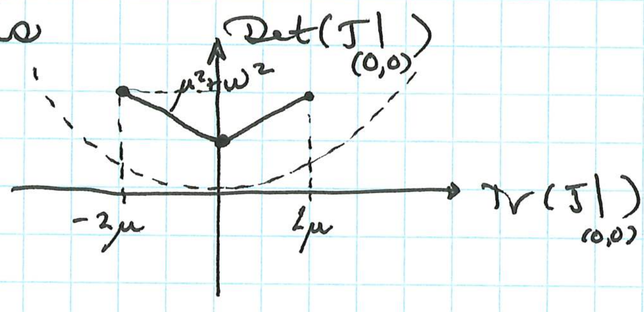
The conditions for a Hopf bifurcation are

- $Re(\lambda(\hat{\mu}))=0$ ✓
- $Im(\lambda(\hat{\mu})) \neq 0$, satisfied for $\omega > 0$
- $\frac{d}{d\mu}\left[Re(\lambda(\mu))\right]\Big|_{\hat{\mu}} \neq 0$, In our case, $\frac{d}{d\mu}\left[Re(\lambda(\mu))\right]\Big|_{\hat{\mu}} = 1$.

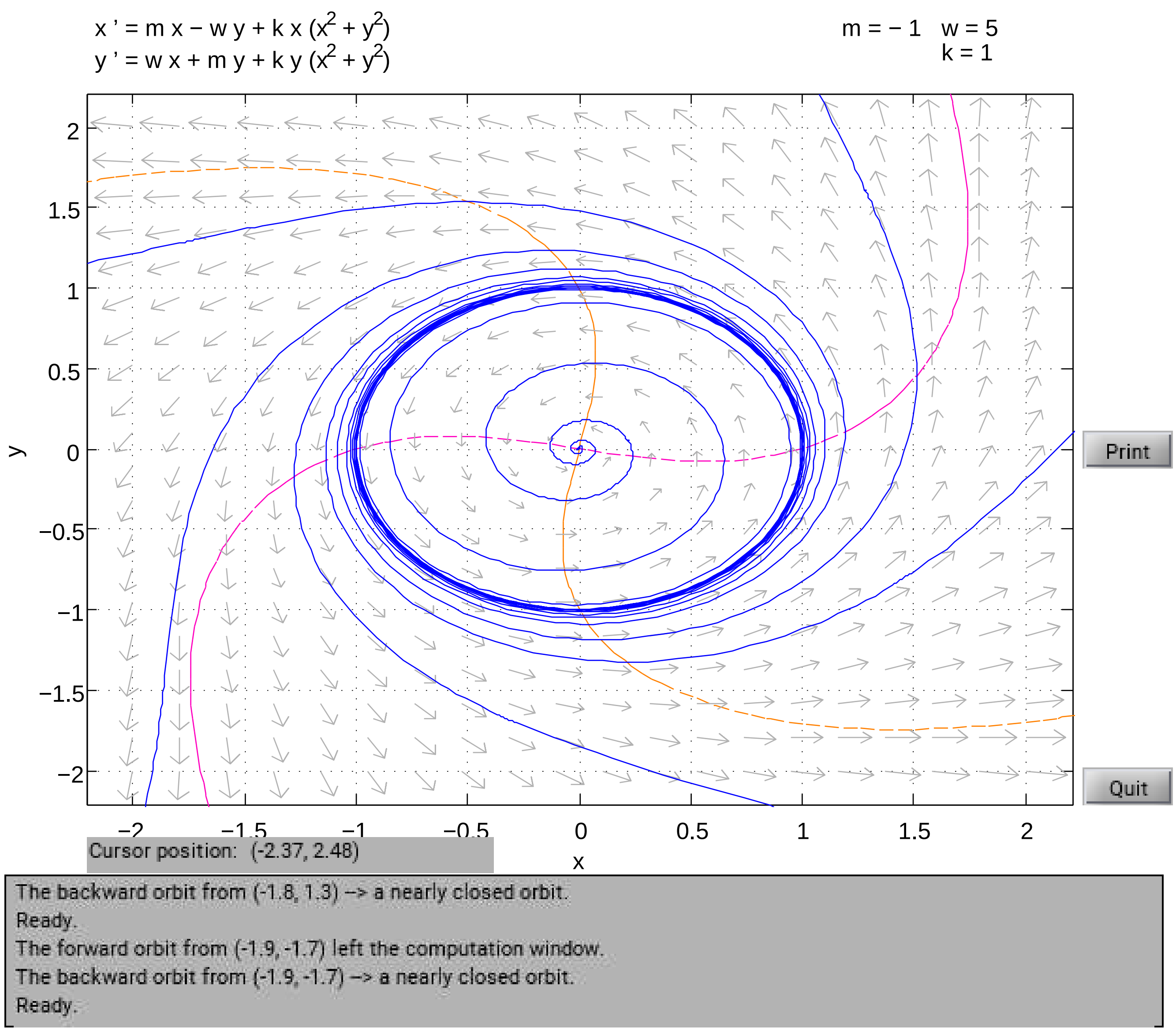
  So this condition is also satisfied.

We thus conclude that there is a Hopf bifurcation at the (0,0) steady state at $\mu=\hat{\mu}=0$. Since the third condition yields a positive derivative, the bifurcation is supercritical.

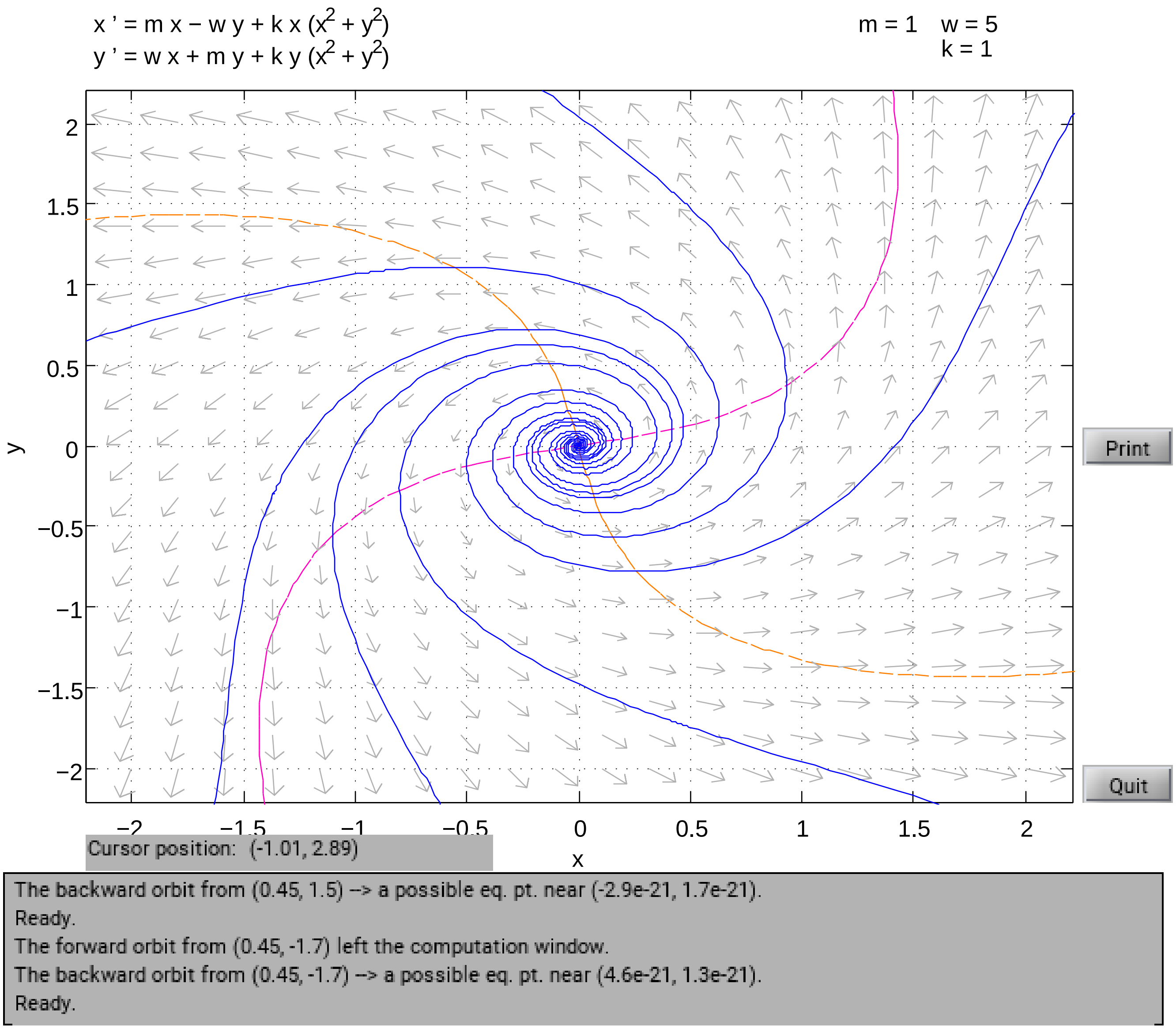




~ phase plane on pplane ~



x ' = m x − w y + k x (x2 + y2)
y ' = w x + m y + k y (x2 + y2)
m = − 1   w = 5
k = 1
y
x
Print
Quit
Cursor position: (-2.37, 2.48)
The backward orbit from (-1.8, 1.3) --> a nearly closed orbit.
Ready.
The forward orbit from (-1.9, -1.7) left the computation window.
The backward orbit from (-1.9, -1.7) --> a nearly closed orbit.
Ready.


x ' = m x − w y + k x (x2 + y2)
y ' = w x + m y + k y (x2 + y2)
m = 1 w = 5
k = 1
Print
Quit
y
x
Cursor position: (-1.01, 2.89)
The backward orbit from (0.45, 1.5) --> a possible eq. pt. near (-2.9e-21, 1.7e-21).
Ready.
The forward orbit from (0.45, -1.7) left the computation window.
The backward orbit from (0.45, -1.7) --> a possible eq. pt. near (4.6e-21, 1.3e-21).
Ready.

Ex 2: Consider the system

$$\begin{cases} \dot{x} = y & = f(x,y) \\ \dot{y} = -x - x^3 + \mu y + k x^2 y & = g(x,y) \end{cases}$$

Show that this system exhibits a Hopf bifurcation.

i) Steady States

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \iff \begin{cases} y = 0 \\ -x - x^3 = 0 \end{cases} \iff \begin{cases} y = 0 \\ x = 0 \text{ or } \underbrace{-1 - x^2 = 0}_{\text{not possible for real } x} \end{cases}$$

∴ There is one steady state at $(0,0)$.

ii) Stability

$$J = \begin{bmatrix} 0 & 1 \\ -1 - 3x^2 + 2kxy & \mu + kx^2 \end{bmatrix}$$

So $J\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$ and the eigenvalues are given by

$$-\lambda(\mu - \lambda) + 1 = 0 \iff \lambda^2 - \mu\lambda + 1 = 0$$

$$\iff \lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

$\lambda_\pm$ are complex for $|\mu| < 2$.
At $\mu = 0$, $\lambda_\pm$ is pure imaginary, and so there is potential for a Hopf bifn.

iii) Hopf bifurcation at $\hat{\mu}=0$?

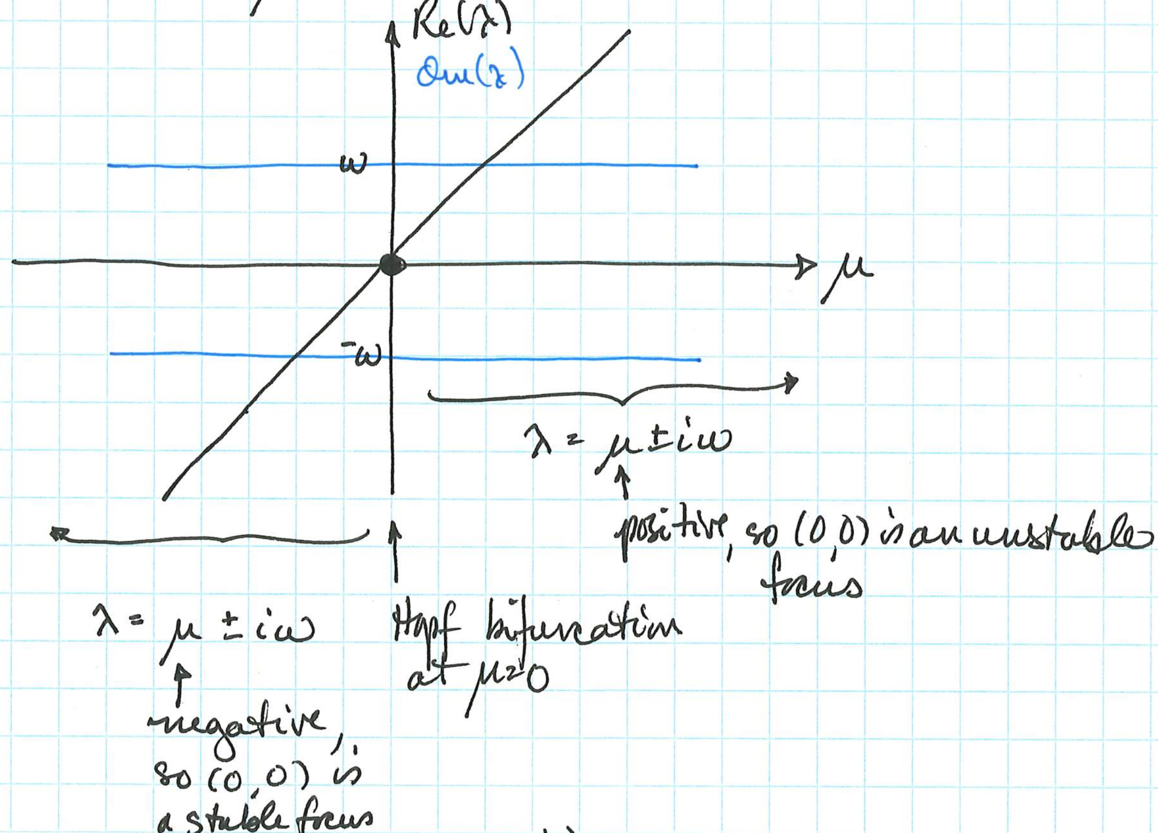
Conditions:

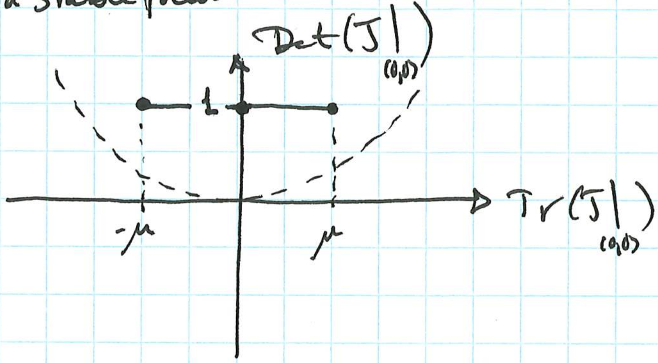
$$Re(\lambda(\hat{\mu})) = \frac{\hat{\mu}}{2} = 0 \quad ✓$$

$$Im(\lambda(\hat{\mu})) = \frac{\sqrt{4}}{2} = 1 \neq 0 \quad ✓$$

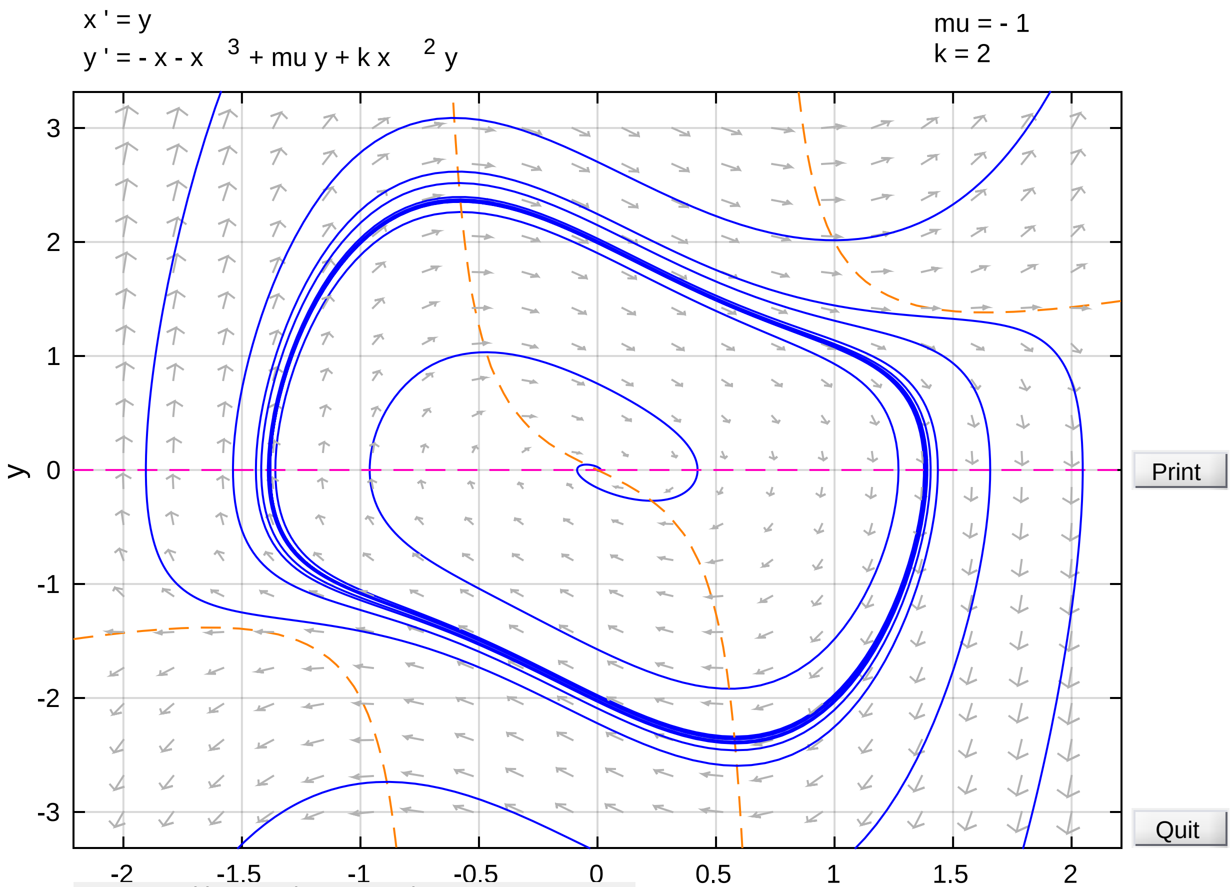
$$\frac{d}{d\mu}\left[Re(\lambda(\mu))\right]\Big|_{\hat{\mu}} = \frac{1}{2} \neq 0 \quad ✓$$

∴ there is a supercritical Hopf bifurcation at the steady state (0,0) when $\mu = \hat{\mu} = 0$.



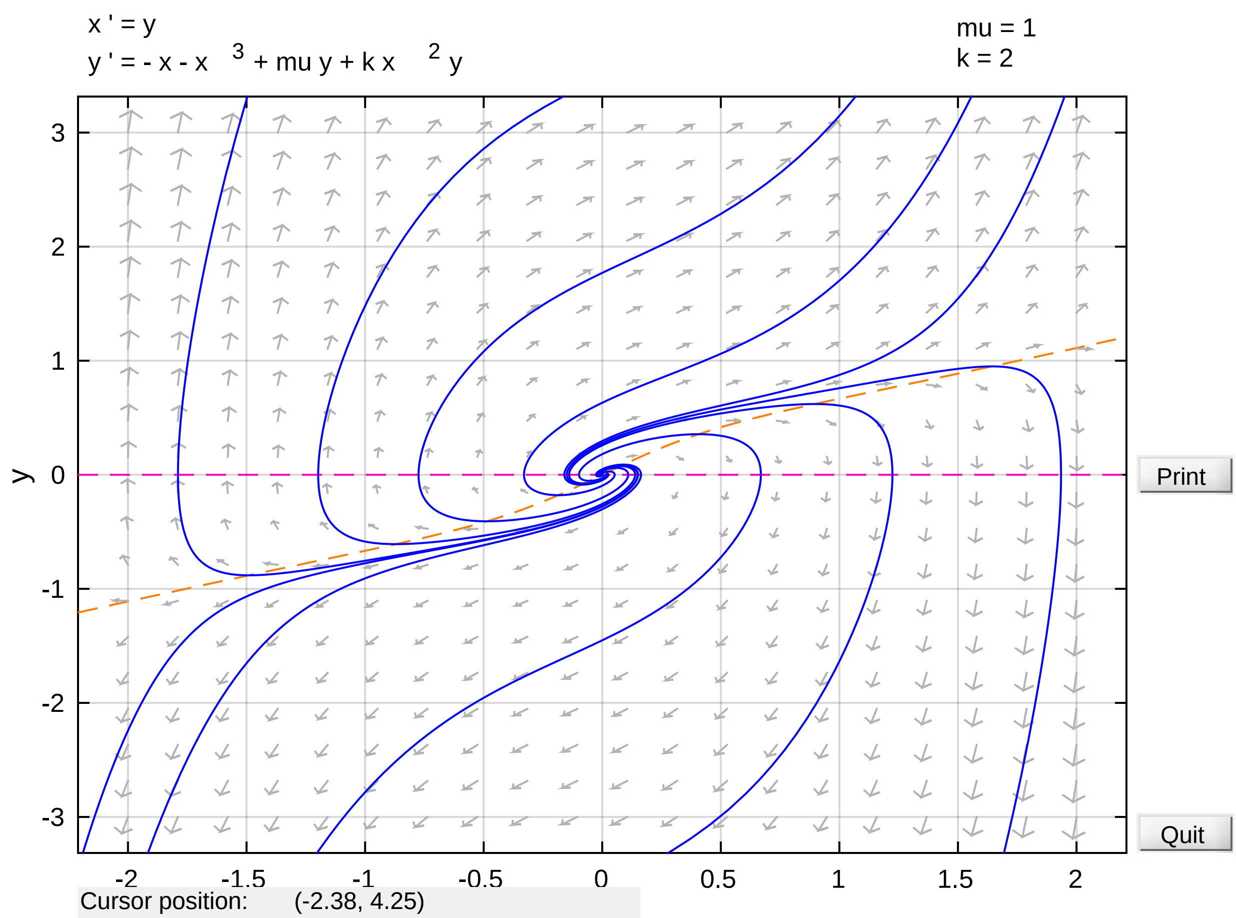


~ phase plane on pplane ~



Cursor position: (-2.42, 4.28)

The backward orbit from (1.6, 2.6) --> a nearly closed orbit.
Ready.
The forward orbit from (1.9, -2.8) left the computation window.
The backward orbit from (1.9, -2.8) --> a nearly closed orbit.
Ready.



The backward orbit from (-1.7, -1.3) --> a possible eq. pt. near (-2.6e-15, 0).
Ready.
The forward orbit from (1.9, -1.5) left the computation window.
The backward orbit from (1.9, -1.5) --> a possible eq. pt. near (4e-15, 0).
Ready.

Ex 3: Consider the system

$$\begin{cases} \dot{x} = x^2 y - x + b, \\ \dot{y} = -x^2 y + a, \end{cases}$$

where x & y are nonnegative, & a & b are in $\mathbb{R}$. Find the steady state & show that the system exhibits a Hopf bifurcation. Illustrate the bifurcation on three different plots. Use either a or b as the bifu param.

Sol'n

i) Steady state $\left(a+b, \frac{a}{(a+b)^2}\right) = (x^*, y^*)$

ii) stability

$$J\Big|_* = \begin{bmatrix} \frac{a-b}{a+b} & (a+b)^2 \\ \frac{-2a}{a+b} & -(a+b)^2 \end{bmatrix}$$

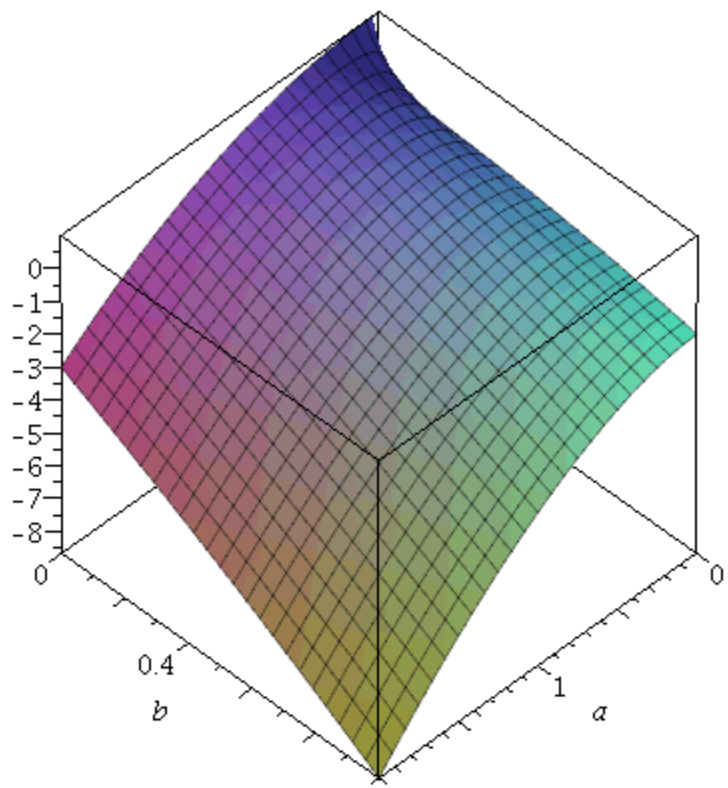
$$\operatorname{Tr} J_* = \frac{a-b}{a+b} - (a+b)^2$$

$$\operatorname{Det} J_* = -(a-b)(a+b) + 2a(a+b) = (a+b)^2$$

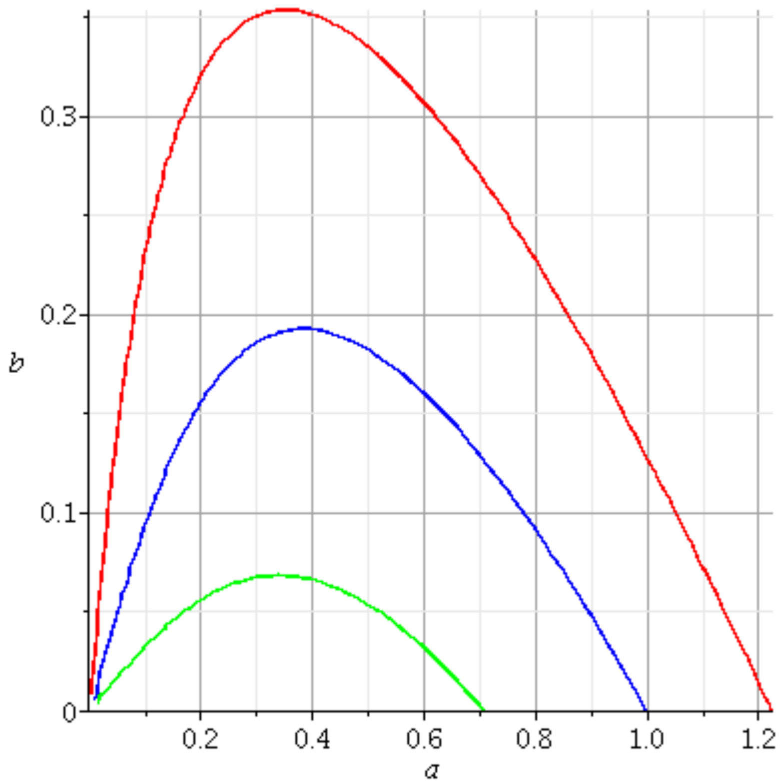
> $TrJ := (a, b) \rightarrow \frac{(a-b)}{a+b} - (a+b)^2; DetJ := (a, b) \rightarrow (a+b)^2;$

$$TrJ := (a, b) \rightarrow \frac{a-b}{a+b} - (a+b)^2$$
$$DetJ := (a, b) \rightarrow (a+b)^2 \tag{2}$$

> $plot3d(\{TrJ(a, b)\}, a = 0..2, b = 0..1);$



```
> implicitplot([TrJ(a, b) = -0.5, TrJ(a, b) = 0, TrJ(a, b) = 0.5], a = 0 ..2, b = 0 ..1, gridlines, colour = [red, blue, green], gridrefine
   = 2);
```



We see that the trace of the Jacobian varies smoothly with a and b, and passes from positive to negative values for fixed a as b increases. We also know that the determinant is positive for all a and b as long as they are not both zero. So there is a subcritical Hopf Bifurcation at the steady state when the point (a,b) lies on the blue curve above and b is increasing. For (a,b) values below the blue curve, the Trace is positive and the steady state is an unstable focus. For (a,b) values above the blue curve the Trace is negative and the steady state is a stable focus.

To illustrate, we set a = 0.4 and plot the Trace of the Jacobian (which is twice the real part of the eigenvalues) as b is increased from 0 to 0.3:.

```
> plot({TrJ(0.4, b)}, b = 0 ..0.3, gridlines);
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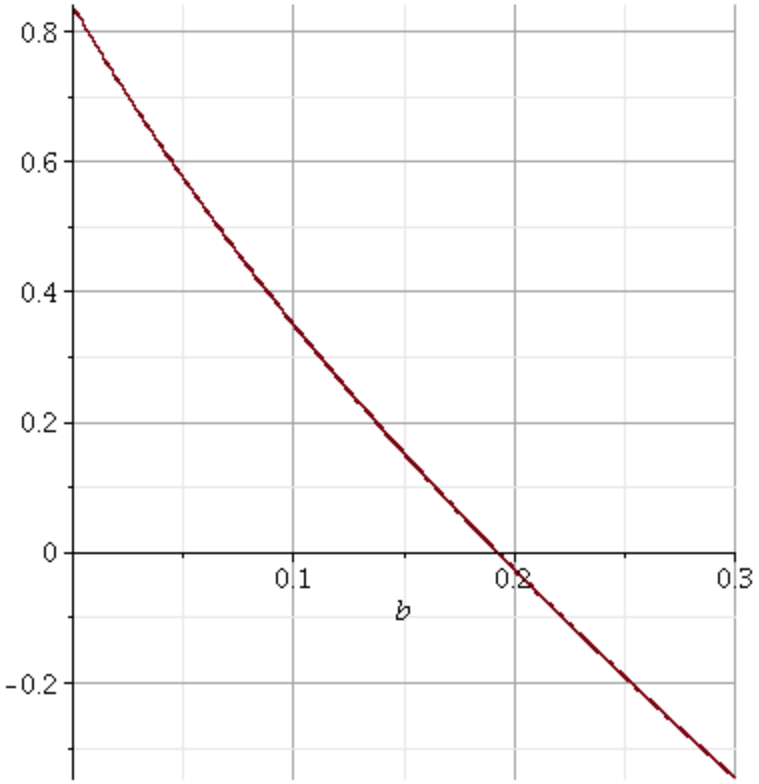


We can also plot the Determinant of the Jacobian as a function of the Trace.

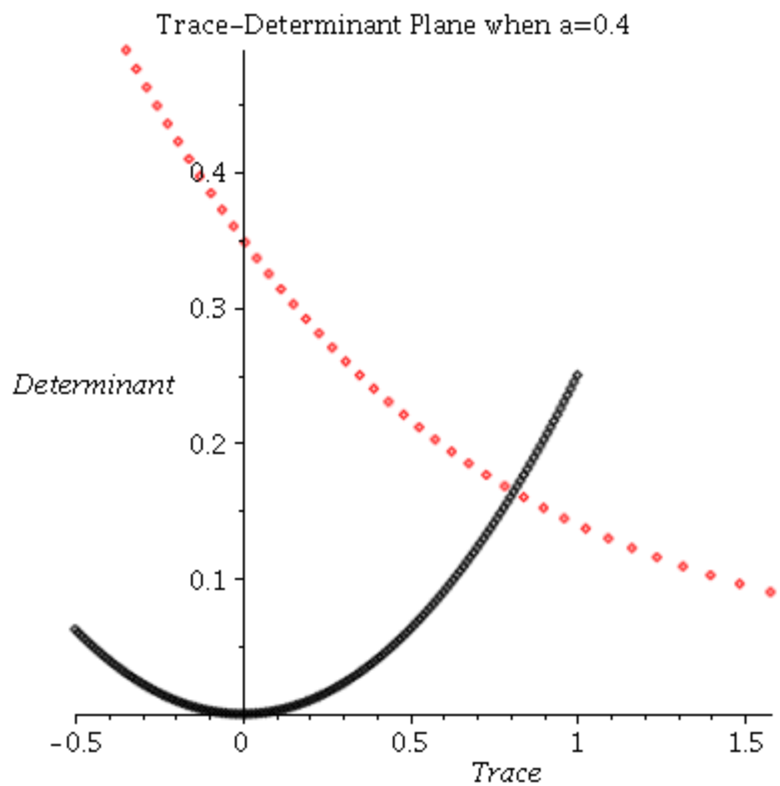
> $P1 := pointplot\left(\left[seq\left(\left[TrJ\left(0.4, \frac{nb}{100}\right), DetJ\left(0.4, \frac{nb}{100}\right)\right], nb = -10..30\right)\right], labels = [Trace, Determinant], colour = [red], title = \text{"Trace - Determinant Plane when a=0.4"}\right);$

$$P1 := PLOT(...) \tag{3}$$

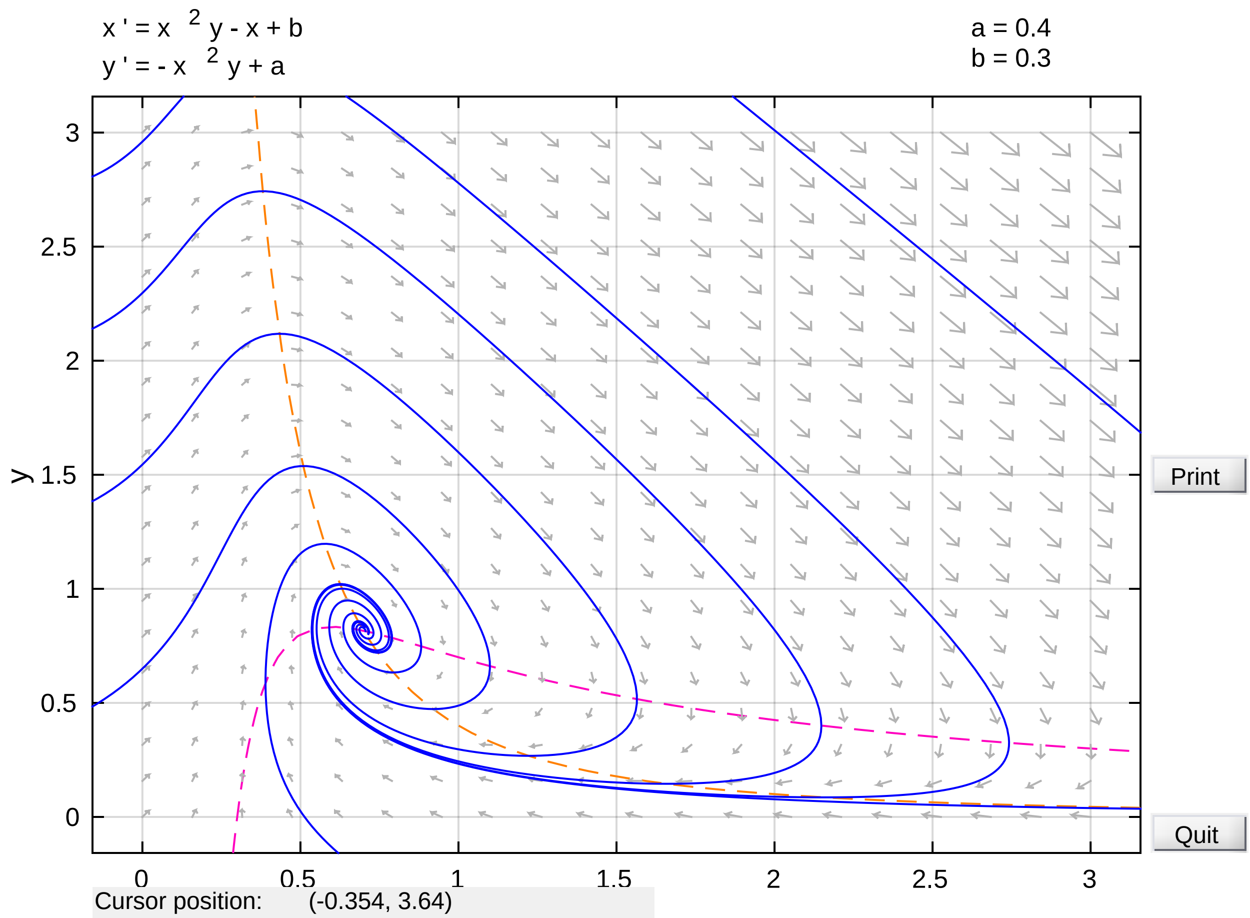
> $P2 := pointplot\left(\left[seq\left(\left[\frac{n}{100}, \frac{1}{4} \cdot \left(\frac{n}{100}\right)^2\right], n = -50..100\right)\right], labels = [Trace, Determinant], colour = [black]\right)$

$$P2 := PLOT(...) \tag{4}$$

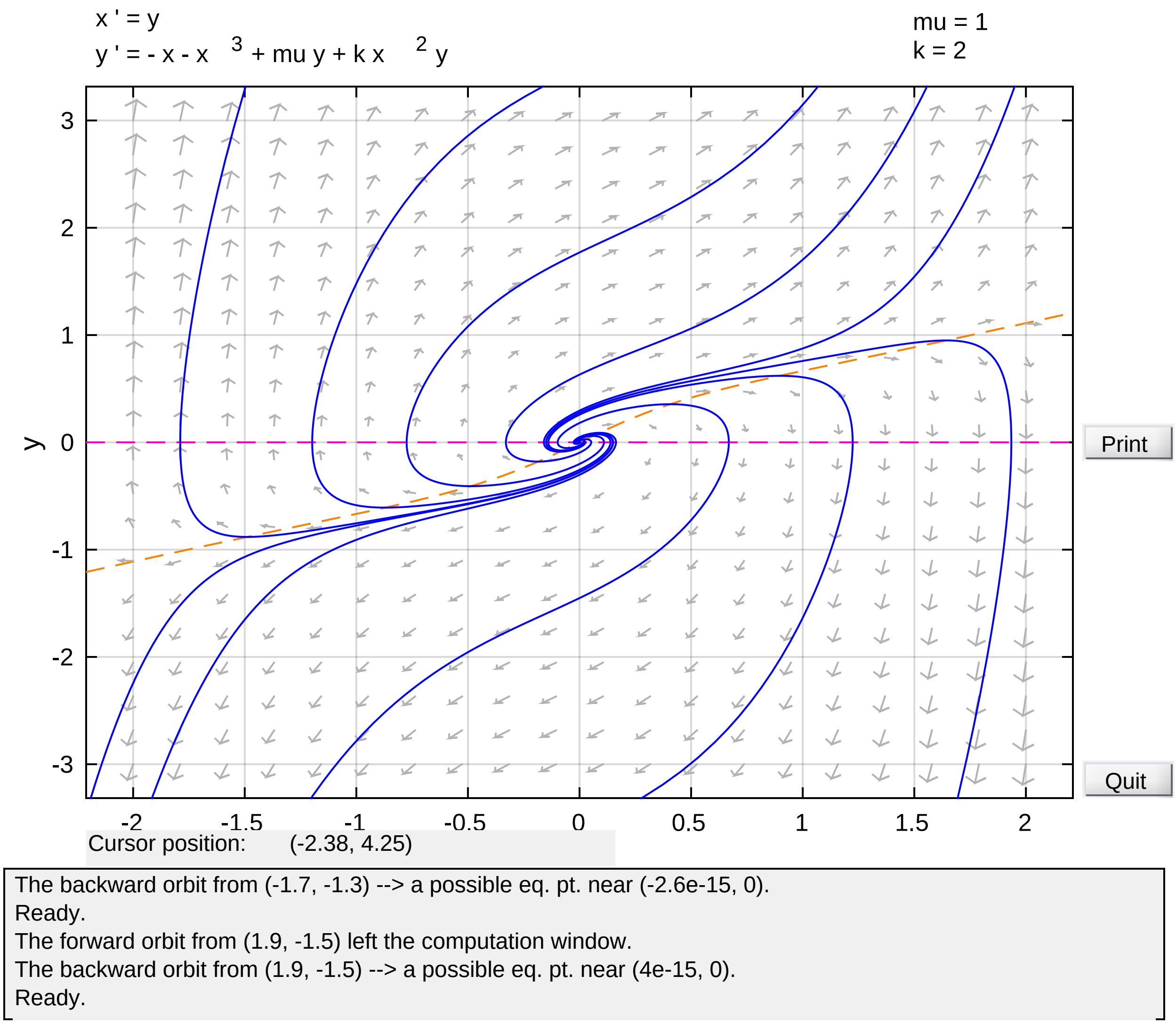
> $display(P1, P2);$



Note that the red dots show the movement of the system in the Trace-Determinant plane as b decreases (Trace is a decreasing function of b - we know this from the plot above). The black dots show the boundary above which the steady state is a focus, and below which it is a node.



The backward orbit from (1.9, 1.1) left the computation window.
Ready.
The forward orbit from (2.4, 2.6) --> a possible eq. pt. near (0.7, 0.82).
The backward orbit from (2.4, 2.6) left the computation window.
Ready.


x ' = y
y ' = - x - x 3 + mu y + k x 2 y
mu = 1
k = 2
3
2
1
0
-1
-2
-3
y
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
Print
Quit
Cursor position: (-2.38, 4.25)
The backward orbit from (-1.7, -1.3) --> a possible eq. pt. near (-2.6e-15, 0).
Ready.
The forward orbit from (1.9, -1.5) left the computation window.
The backward orbit from (1.9, -1.5) --> a possible eq. pt. near (4e-15, 0).
Ready.