Additional Concepts that we will need for Thursday's class

Change of area or volume by the flow

Consider the system (fr a#5)

$$\begin{cases} \dot{x} = -y + \left(\frac{x^3}{3} - x\right) \\ \dot{y} = x \end{cases} \qquad (1)$$

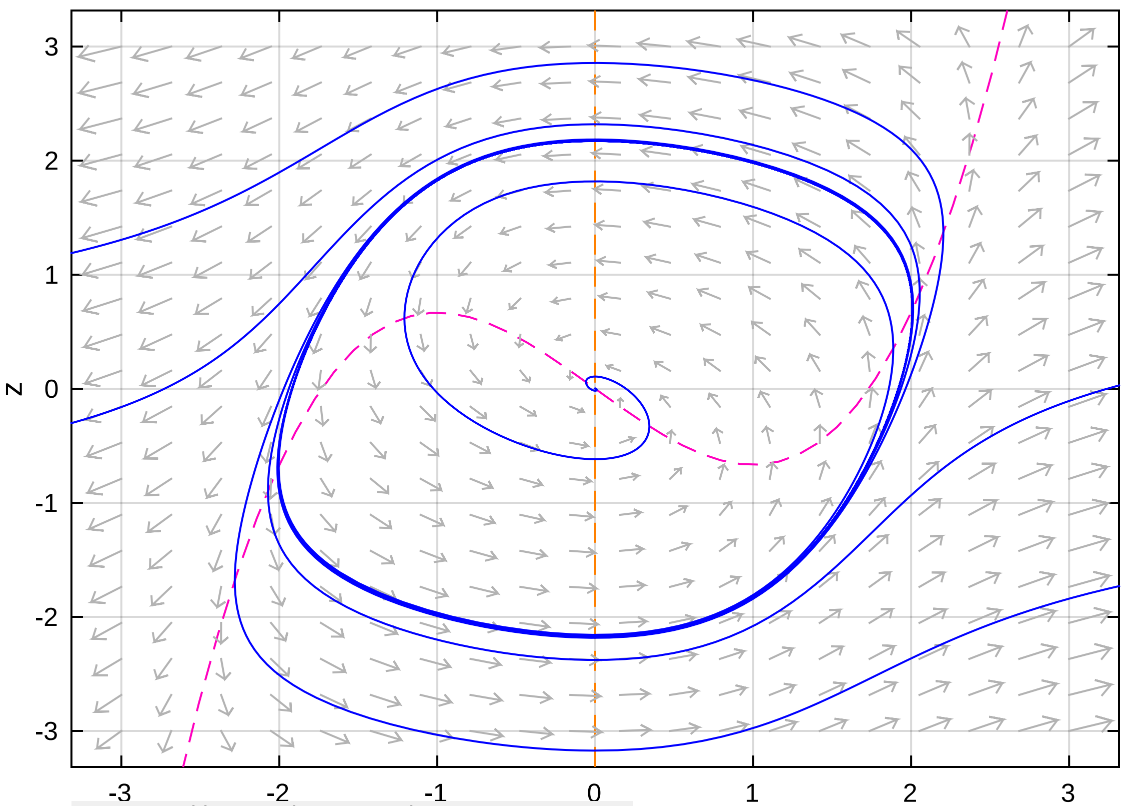
which we can write

$$\dot{\vec{x}} = \vec{F}(\vec{x}) \qquad (2)$$

What would happen to a shape, such as a square, if it were carried by the flow?

- look at the phase plane & make guesses
- then do the experiment

$x' = -z + (x^3/3 - x)$
$z' = x$

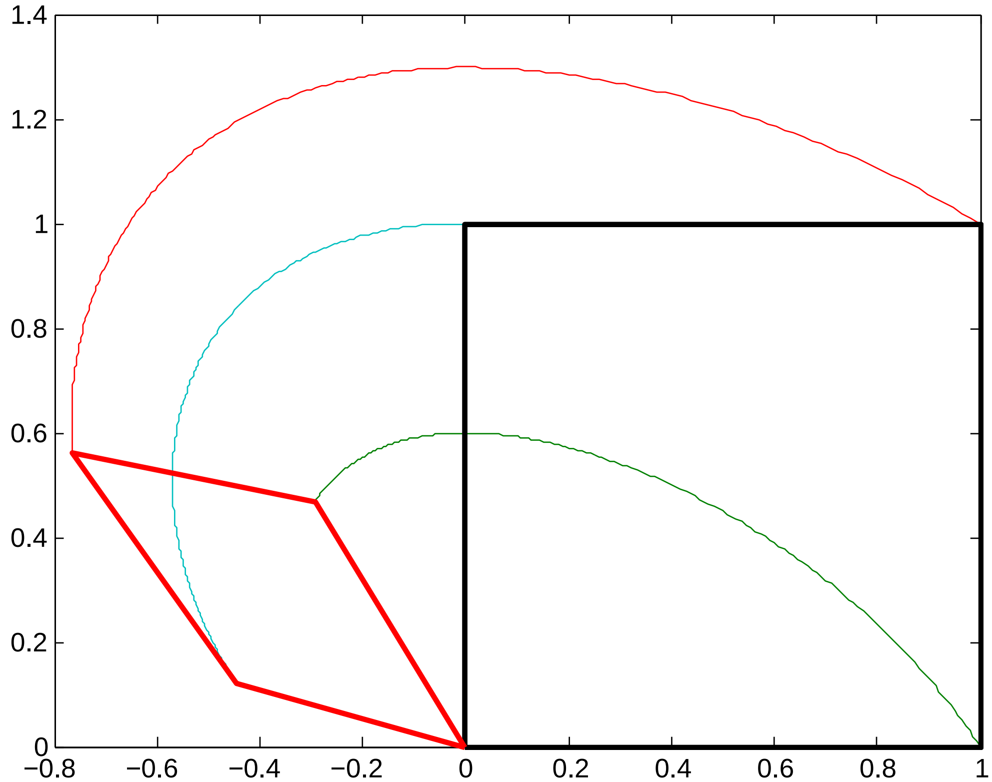


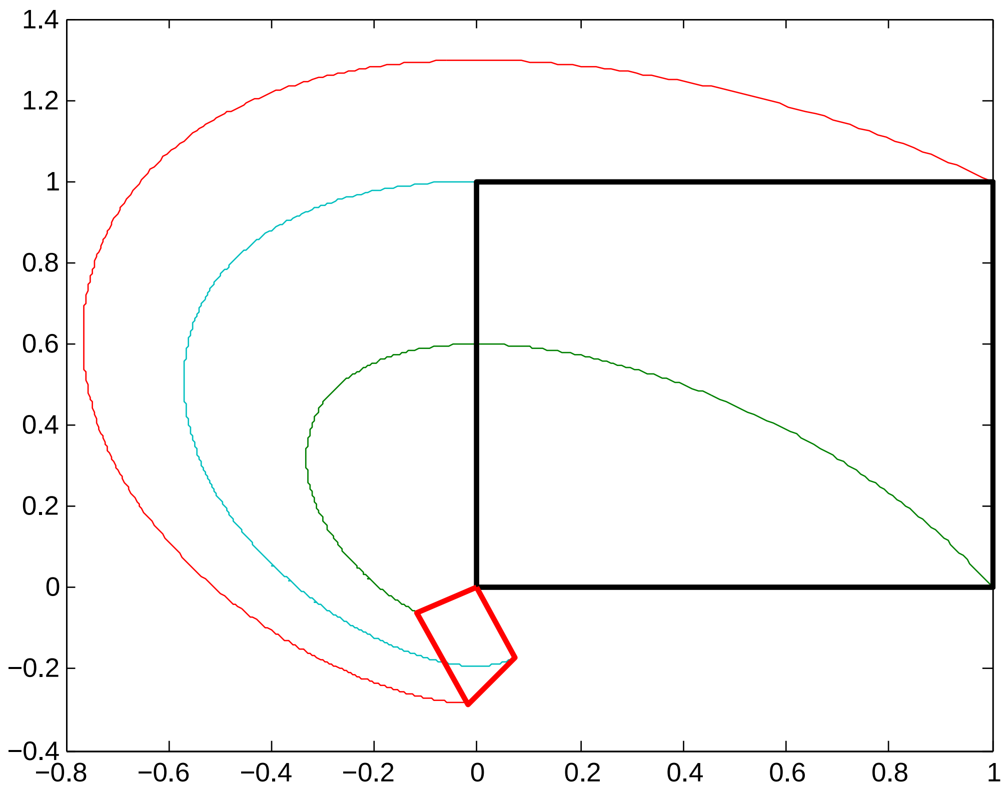
Print

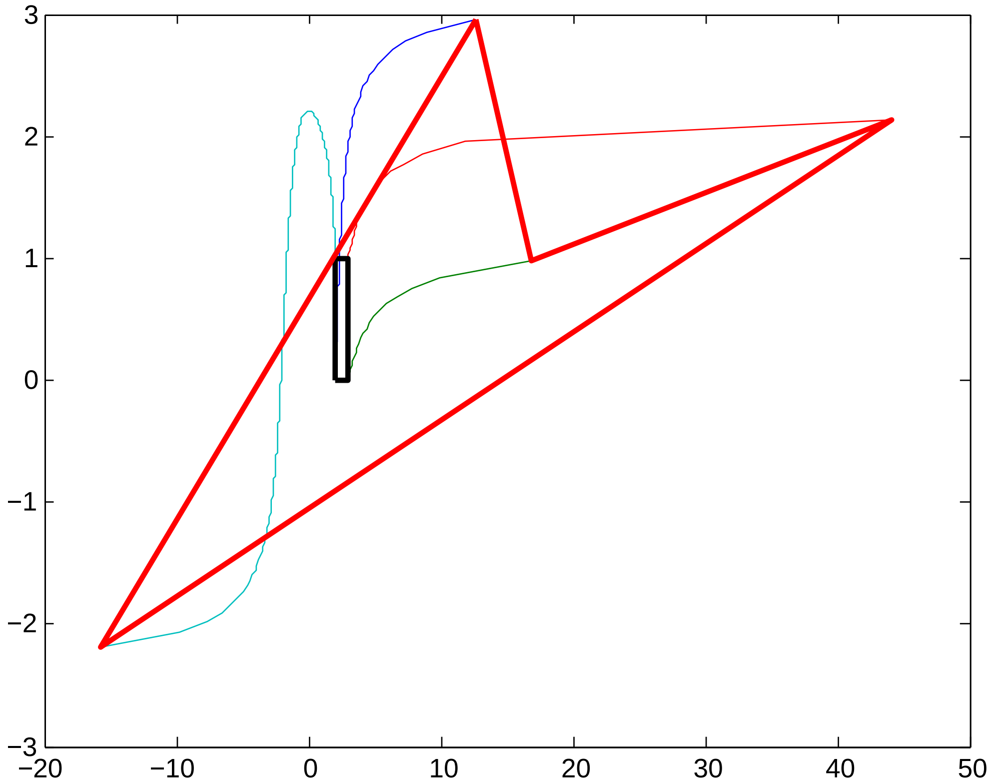
Quit

Cursor position: (-3.97, 4.26)

The backward orbit from (2.9, -0.15) --> a nearly closed orbit.
Ready.
The forward orbit from (-2.7, 0.0088) left the computation window.
The backward orbit from (-2.7, 0.0088) --> a nearly closed orbit.
Ready.


1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



3
2
1
0
−1
−2
−3
−20
−10
0
10
20
30
40
50

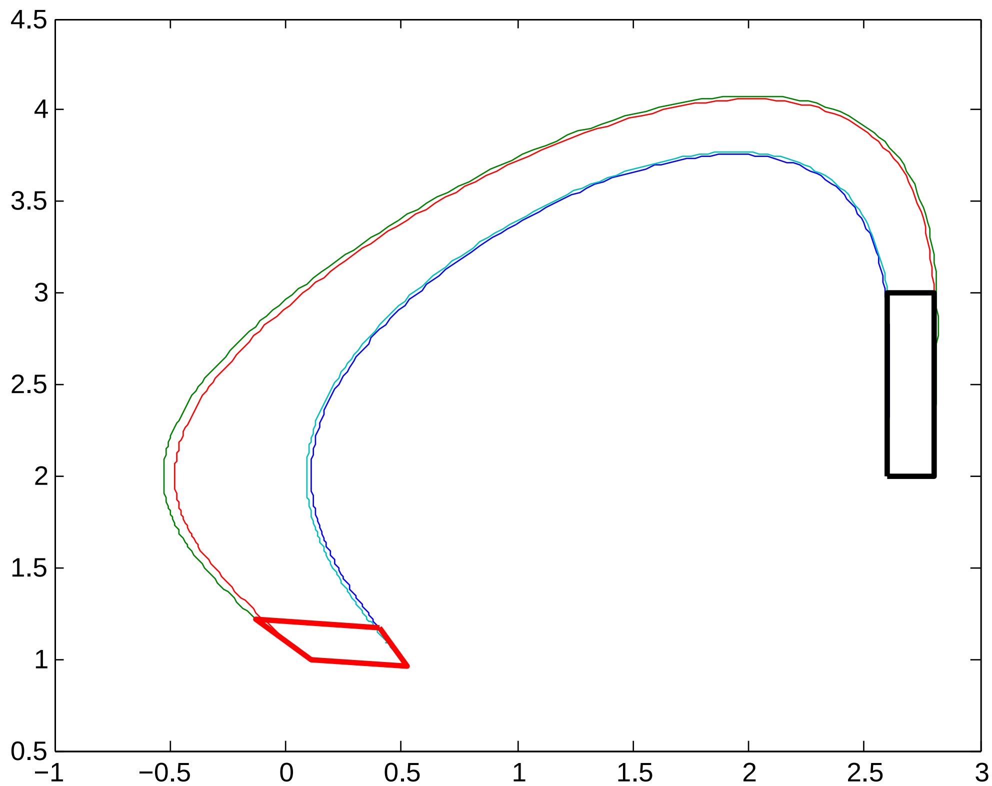
General observations

1. The initial square is distorted.
2. For objects that start inside the limit cycle, the area decreases with time.
3. For objects that start outside the limit cycle, the area increases with time.

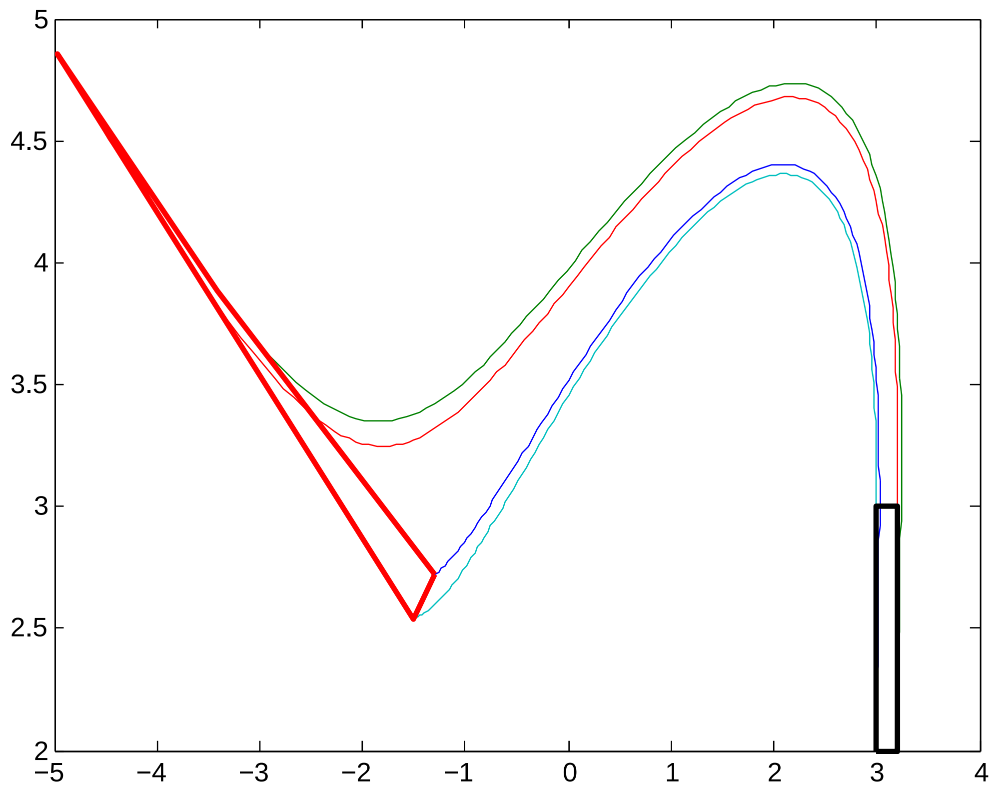
Consider another system (for a#5)

$$\begin{cases} \dot{x} = (x-y)(y-2) \\ \dot{y} = x^2 - y \end{cases}$$

Predict which parts of the flow are expanding, and which are contracting.



4.5
4
3.5
3
2.5
2
1.5
1
0.5
−1
−0.5
0
0.5
1
1.5
2
2.5
3


5
4.5
4
3.5
3
2.5
2
−5
−4
−3
−2
−1
0
1
2
3
4

Theorem

Let $\dot{\vec{x}} = \vec{F}(\vec{x})$ be a system of differential equations in $\mathbb{R}^n$, with flow $\phi(t;\vec{x})$. Let $\mathcal{D}$ be a region in $\mathbb{R}^n$ with (i) finite n-volume and (ii) smooth boundary $\partial\mathcal{D}$. Let $\mathcal{D}(t)$ be the region formed by flowing along for time t,

$$\mathcal{D}(t) = \{ \phi(t;\vec{x}_0) : \vec{x}_0 \in \mathcal{D} \}.$$

a) Let $V(t)$ be the n-volume of $\mathcal{D}(t)$. Then

$$\frac{d}{dt} V(t) = \int_{\mathcal{D}(t)} \underbrace{\vec{\nabla}\cdot\vec{F}(\vec{x})}_{\text{divergence of } \vec{F} \text{ at } \vec{x}} \, dV$$

volume element $dV = dx_1 dx_2 \cdots dx_n$ (area in $n=2$)

b) If the divergence of $\vec{F}$ is a constant (independent of $\vec{x}$), then

$$V(t) = V(0) e^{(\vec{\nabla}\cdot\vec{F})t} .$$

Thus, if the vector (flow) field is divergence free, with $\vec{\nabla}\cdot\vec{F} \equiv 0$, then the volume is preserved. If $\vec{\nabla}\cdot\vec{F} > 0$, then the volume grows exponentially, & " $\vec{\nabla}\cdot\vec{F} < 0$, " " " decays " .

Poincaré Maps

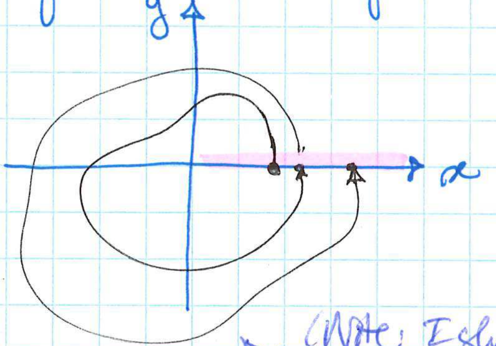
Consider

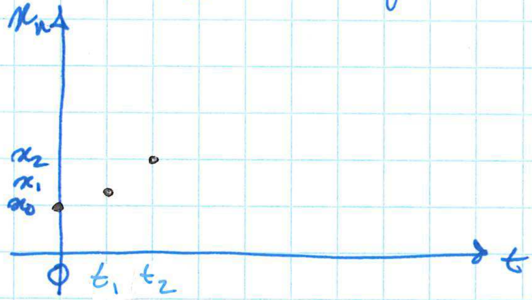
$$\begin{cases} \dot{x} = y + x(1-x^2-y^2) \\ \dot{y} = -x + y(1-x^2-y^2) \end{cases} \quad \ldots\ldots (1)$$

In polar coordinates, (1) becomes

$$\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = -1 \end{cases} \quad \ldots\ldots\ldots (2)$$

We are interested in the first return of trajectories from the half line $\{(x,0): x>0\}$ to itself.

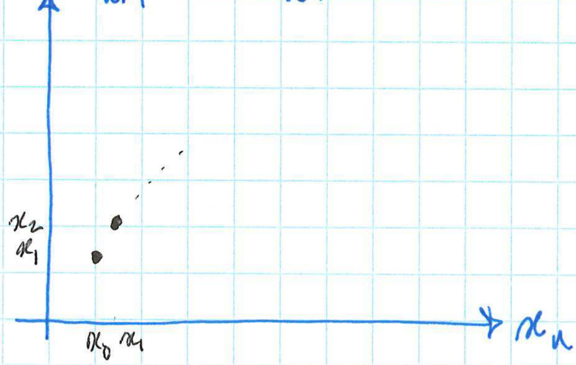




(Note: I should have sketched these trajectories rotating in the opposite direction.)

First return map

$$x_{n+1} = P(x_n)$$



We can derive $P(x_n)$ for this system, using (2):

$$\frac{d\theta}{dt} = -1 \Leftrightarrow \theta = -t + \theta_0$$

We set $\theta_0 = 0$, & observe that θ is periodic. So $x_1 = x(2\pi)$, $x_2 = x(4\pi)$, ... Similarly for r.

$$\frac{dr}{dt} = r(1-r^2) \Leftrightarrow \int_{r_0}^{r(t)} \frac{1}{r(1-r^2)}\,dr = \int_0^t dt$$

$$\Leftrightarrow \int_{r_0}^{r(t)} \left(\frac{1}{r} + \frac{1}{2(1-r)} - \frac{1}{2(1+r)}\right) dr = \int_0^t dt$$

$$\Leftrightarrow \ln\left(\frac{r(t)^2}{1-r(t)^2}\right) - \ln\left(\frac{r_0^3}{1-r_0^2}\right) = 2t$$

Now solve for $r(t)$:

$$r(t)^2 = \frac{r_0^2 e^{2t}}{1-r_0^2 + r_0^2 e^{2t}} = \frac{1}{1 + e^{-2t}(r_0^{-2}-1)}.$$

$$\therefore\ r(t) = \left[1 + e^{-2t}(r_0^{-2}-1)\right]^{-\frac{1}{2}} \text{ OR } = r_0\left[r_0^2 + e^{-2t}(1-r_0^2)\right]^{-\frac{1}{2}} \quad \ldots (3)$$

Since it takes a length of time $t = 2\pi$ to return to the half line $\{(x,0) : x > 0\}$, we have

$$r_1 = r(2\pi) = \left[1 + e^{-4\pi}(r_0^{-2}-1)\right]^{-1/2} = P(r_0) \quad \ldots (4)$$

$$\text{OR } = r_0\left[r_0^2 + e^{-4\pi}(1-r_0^2)\right]^{-1/2} \quad \ldots (5)$$

Observe that:

a) $r_1 = r_0$ if $r_0 = 0$ or 1 (these points "return to themselves")

b) if $r > 1$

$$P(r) = [1 + e^{-4\pi}\underbrace{(r^{-2}-1)}_{<0}]^{-\frac{1}{2}}$$

$$\underbrace{\quad\quad\quad\quad\quad\quad\quad}_{<1}$$

$\therefore P(r) > 1$

Also

$$P(r) = r\left[r^2 + e^{-4\pi}(1-r^2)\right]^{-\frac{1}{2}}$$

$$= r\left[\underbrace{1 - (1-e^{-4\pi})\underbrace{(1-r^2)}_{<0}}_{>1}\right]^{-\frac{1}{2}}$$

$\therefore P(r) < r$

So we have

$$1 < P(r) < r \quad \ldots\ldots \quad (6)$$

We also have

$$\frac{dP}{dr} = \frac{e^{-4\pi}}{[r^2 + e^{-4\pi}(1-r^2)]^{3/2}} = \frac{e^{-4\pi}}{r^3}(P(r))^3 > 0$$

$$\ldots (7)$$

$\therefore$ (6) + (7) together tell us that $\lim_{n\to\infty} r_n = 1$ ... (8)

c) if $r < 1$

We obtain $0 < r < P(r) < 1$ . . . . (9).

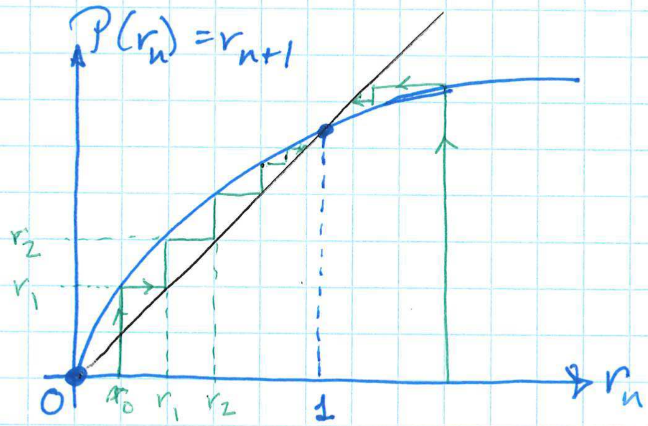
Results (9) with (7) tell us that (8) still holds.

Summary

if $r_0 = 0$, $\lim_{n\to\infty} r_n = 0$

$r_0 > 0$, $\lim_{n\to\infty} r_n = 1$

We can see this graphically:



"cobwebbing"