Lecture #2

Opinion Dynamics Model

$$(1)\ \ldots\ \begin{cases} \dot{L}_2 = L_1[L_2 + p_a L_1] - L_2[1 - L_2 - p_a L_1] \\ \dot{L}_1 = L_2[1 - L_2 - p_a L_1] + R_1[L_2 + L_1] - L_1[1 - (1-p_a)L_1] \\ \dot{R}_1 = L_1[R_1 + R_2] + R_2[1 - R_2 - p_a R_1] - R_1[1 - (1-p_a)R_1] \\ \dot{R}_2 = R_1[R_2 + p_a R_1] - R_2[1 - R_2 - p_a R_1] \end{cases}$$

(page 4 of Baumgaertner et al (2018))

Simulations....

(figure 4, p4 of Baumgaertner et al (2018))

What do we see? (IC gives a small advantage to the L side)

step 1: L_2 & $R_2 \to 0$, L_1 & $R_1 \to \frac{1}{2}$ } centering, symmetric

step 2: $L_2 \to 1$, L_1 also goes to zero, $R_1 \to 0$ & $R_2 \to 0$ } consensus

Proposed Simplification:

step 1: Set $\begin{cases} L_2 = R_2 = y \\ L_1 = R_1 = x \end{cases}$

Step 2: Set $\begin{cases} R_1 = R_2 = 0 \\ L_1 = x, \ L_2 = y \end{cases}$

Step 1: Simplified Model for <u>Centering</u>

$$\dot{y} = x[y + p_a x] - y[1 - y - p_a x]$$

$$\dot{x} = y[1 - y - p_a x] + x[y + x] - x[1 - (1 - p_a)x]$$

$$\dot{x} = x[x + y] + y[1 - y - p_a x] - x[1 - (1 - p_a)x]$$

$$\dot{y} = x[y + p_a x] - y[1 - y - p_a x]$$

So we arrive at

$$\begin{cases} \dot{x} = x(y + x) + y(1 - y - p_a x) - x(1 - (1 - p_a)x) \\ \dot{y} = x(y + p_a x) - y(1 - y - p_a x) \end{cases} \quad \ldots (2)$$

How do we analyse this model?

- autonomous $\Rightarrow$ look @ the <u>phase plane</u>

Analysis of Nonlinear ODE System

I) Nullclines and Steady States

Nullclines: Curves along which either $\dot{x}=0$ or $\dot{y}=0$.

Steady States: Also called "equilibrium points", these are points at which the system does not change: $\dot{x}=\dot{y}=0$.

a) Find the nullclines

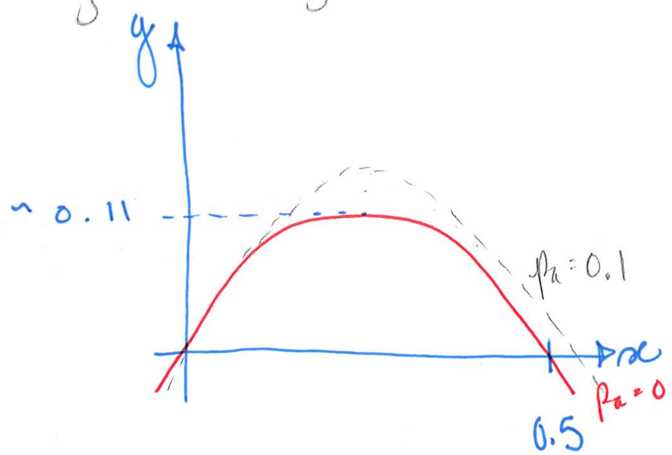
$$\dot{x}=0 \iff x(y+x)+y(1-y-p_a x)-x(1-(1-p_a)x)=0$$

$$\iff xy+x^2+y-y^2-p_a xy-x+(1-p_a)x^2=0$$

$$\iff -y^2+(x+1-p_a x)y+(x^2-x+(1-p_a)x^2)=0$$

$$\iff y^2-(1+(1-p_a)x)y-((2-p_a)x^2-x)=0$$

$p_a=0$ case: $y^2-(1+x)y-(2x-1)x=0$



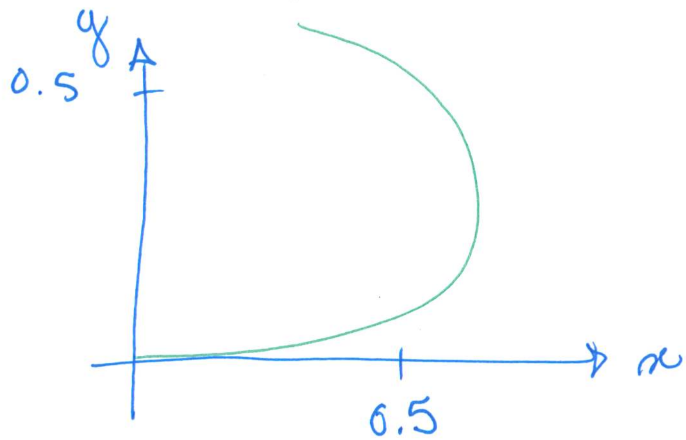
(use Maple for complicated fns)

$$\dot{y}=0 \iff x(y+p_a x)-y(1-y-p_a x)=0$$

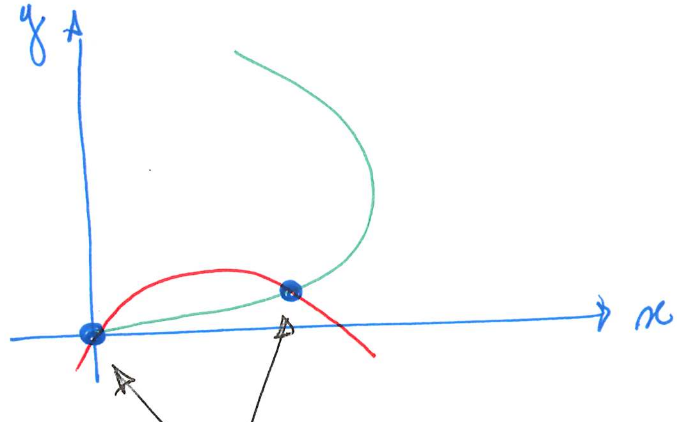
$$\iff xy+p_a x^2-y+y^2+p_a xy=0$$

$$\iff y^2+(x-1+p_a x)y+p_a x^2=0$$

$p_a=0$ case: $y^2+(x-1)y=0$



b) Together:

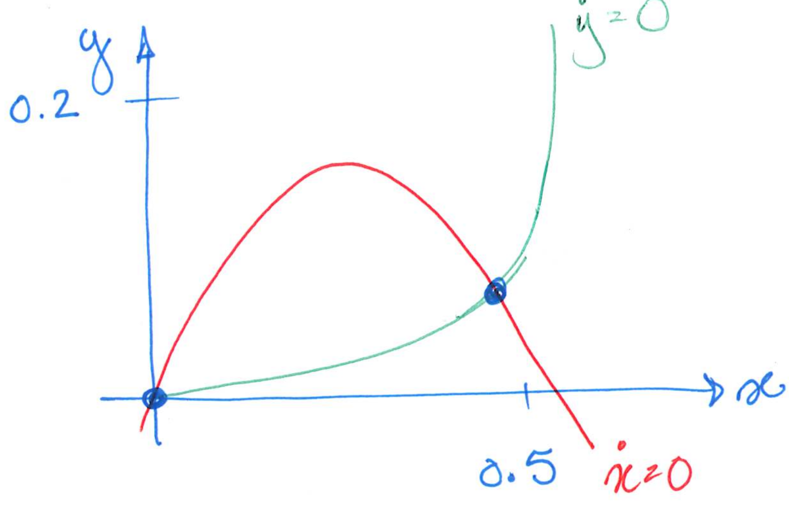


These points are steady states.

II) Determine the Stability of the Steady States

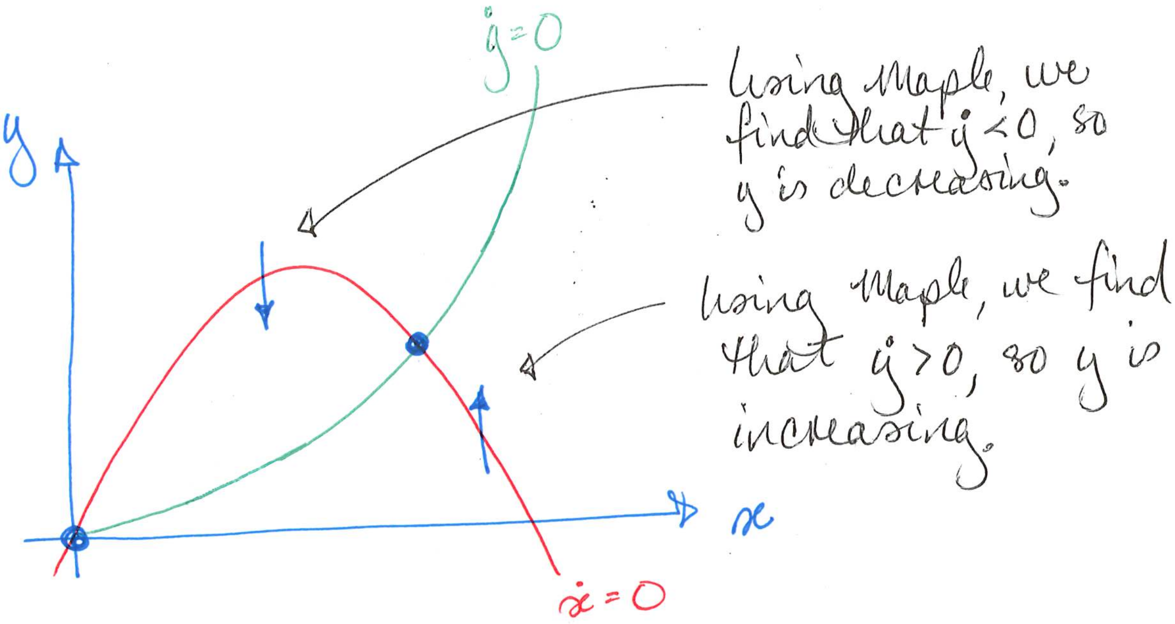
Two methods
a) graphical
b) analytical

Let's start with (a):

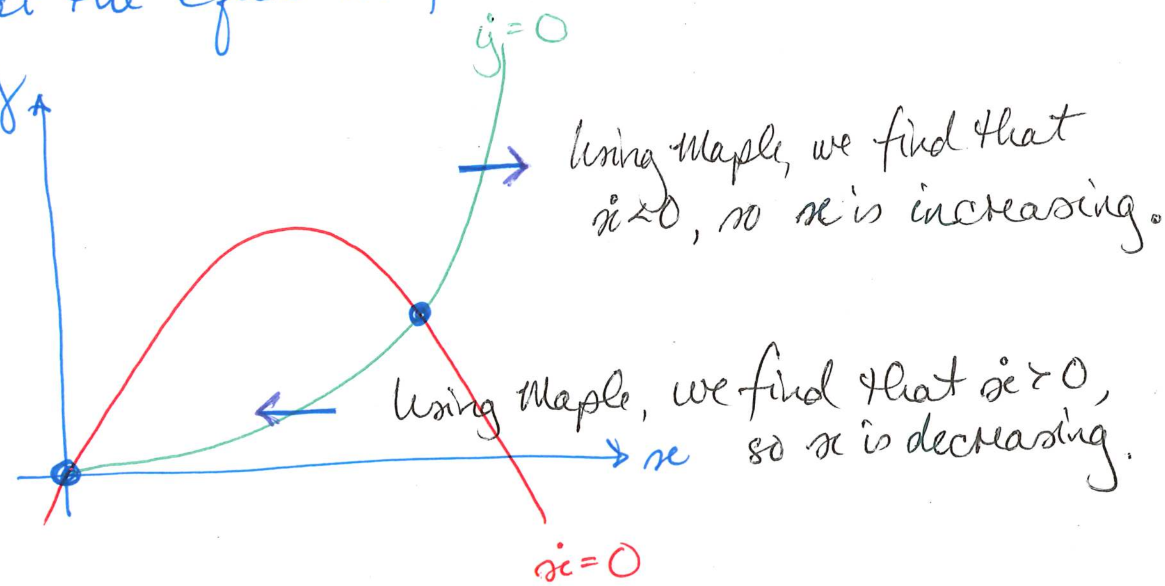


$(p_a = 0.1)$

On the x-nullcline ($\dot{x}=0$), only y changes. We draw arrows to indicate if y is increasing ($\uparrow$) or decreasing ($\downarrow$). To determine if y is increasing or decreasing, we look at the equation for $\dot{y}$ in (2):



Repeat for points on the y-nullcline ($\dot{y}=0$). On this nullcline, only x changes. We draw arrows to indicate if x is increasing ($\rightarrow$) or decreasing ($\leftarrow$). To determine if x is increasing or decreasing, we look at the equation for $\dot{x}$ in (2):



Put it all together on one plot...

(See the Maple output on the website, lecture #2)

Exercises for the Students:

Find the nullclines & steady states, & plot the phase plane diagrams for the following nonlinear systems:

1) $\begin{cases} \dot{x} = x^2 - y \\ \dot{y} = 2xy - 1 \end{cases}$

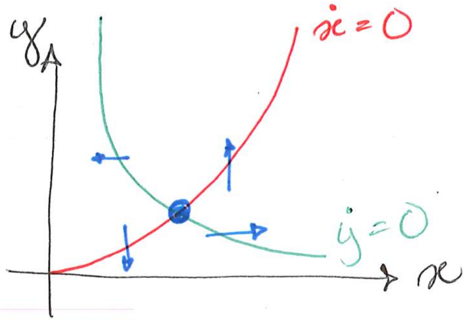
2) $\begin{cases} \dot{x} = x^2 - xy \\ \dot{y} = x + y - 1 \end{cases}$

3) $\begin{cases} \dot{x} = x \sin(y) \\ \dot{y} = x^3 - y \end{cases}$

Answers

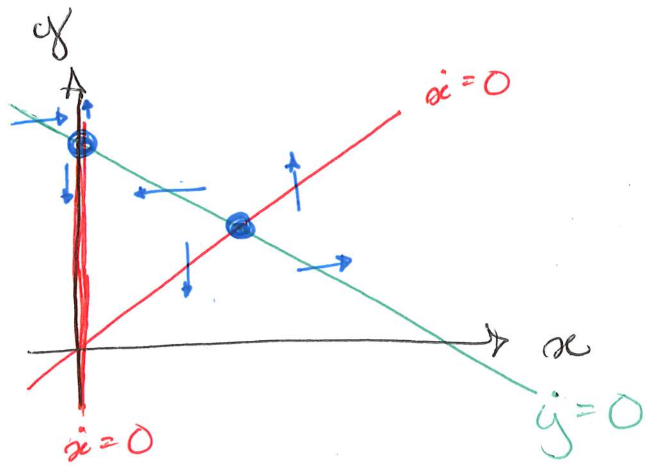
1) $\dot{x} = 0 \iff x^2 - y = 0 \iff y = x^2$

$\dot{y} = 0 \iff 2xy - 1 = 0 \iff y = \dfrac{1}{2x}$



2) $\dot{x} = 0 \iff x^2 - xy = 0 \iff x = 0$ or $x = y$

$\dot{y} = 0 \iff x + y - 1 = 0 \iff y = -x + 1$



3) $\begin{cases} \dot{x} = 0 \iff x\sin(y) = 0 \iff x = 0 \text{ or } y = n\pi,\ n \in \mathbb{Z} \\ \dot{y} = 0 \iff y = x^3 \end{cases}$