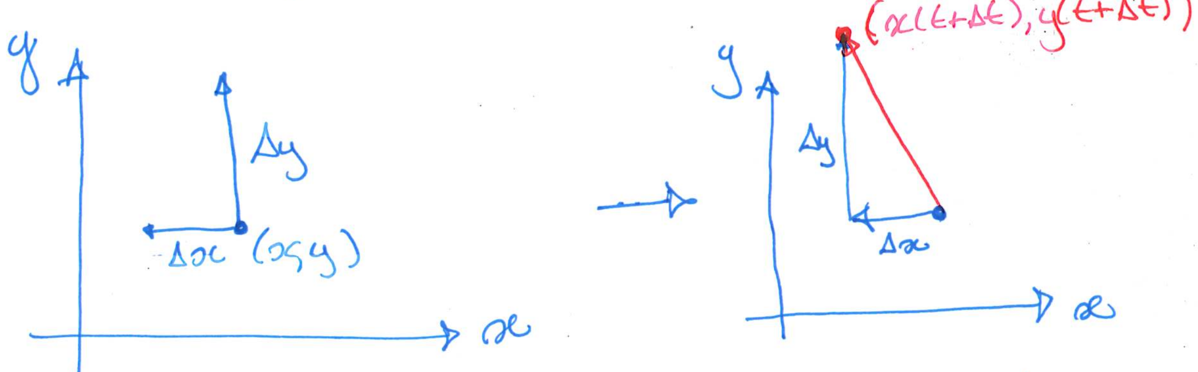
Lecture #3

Consider the two dimensional autonomous ODE system

$$(3) \quad \cdots \begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{\Delta x}{\Delta t} = f(x,y) \\ \dfrac{\Delta y}{\Delta t} = g(x,y) \end{cases} \cdots (4)$$

We observed that we can evaluate $f(x,y)$ & $g(x,y)$ at any point (x,y) and draw an arrow representing the movement of the solution in the phase plane over a small interval Δt:



For small enough Δt, & repeating this procedure, we can trace the solution $(x(t), y(t))$ in the phase plane.

Vector Function

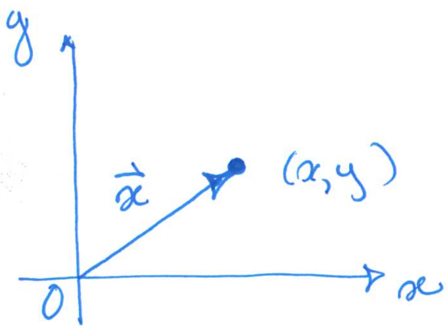
A Summary of Facts about Vector Functions (from Calculus)

1. The pair $(x(t), y(t))$ represents a curve in the xy plane with t as a parameter.
2. $\vec{x}(t) = (x(t), y(t))$ also represents a position vector: a vector with tail at $(0,0)$ and head at $(x(t), y(t))$.
3. The vector $d\vec{x}/dt$, which is just the pair $(dx/dt, dy/dt)$ has a well-defined geometric meaning. It is a vector that is tangent to the solution curve at $\vec{x}(t)$. Its magnitude, written $|d\vec{x}/dt|$, represents the speed of motion of the point $\vec{x}(t) = (x(t), y(t))$ along the curve.
4. The set of equations (3) can be written in vector form

$$\dot{\vec{x}} = \frac{d\vec{x}}{dt} = \vec{F}(\vec{x}). \qquad (5)$$

from Mathematical Models in Biology by L. Edelstein-Keshet (2005) SIAM

Here, the vector function $\vec{F} = (f, g)$, assigns a vector to every location $\vec{x}$ in the plane; $\vec{x}$ is the position vector and $\dot{\vec{x}}$ is the velocity vector.



(x,y) is a point on the solution curve
$\vec{x}$ is a vector that points to (x,y)

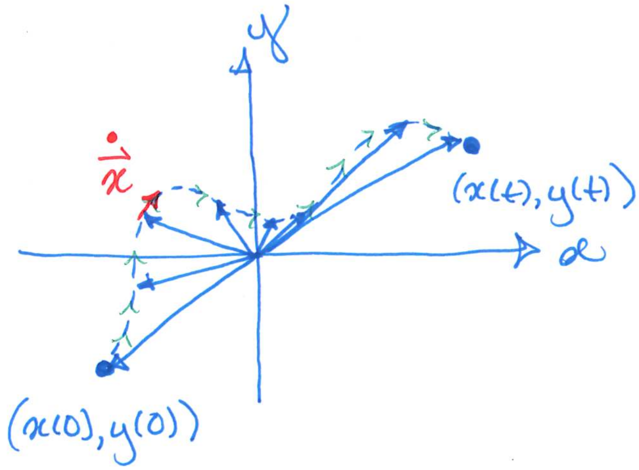


Illustration of a sequence of position vectors $\vec{x}$ tracking the solution $(x(t), y(t))$, and one tangent vector, $\dot{\vec{x}}$.

Nullclines

The nullclines are loci for which

1. $\dot{x} = 0$ (the x nullcline); that is, $f(x,y) = 0$. On these curves, direction vectors are vertical ($\uparrow$ or $\downarrow$)

2. $\dot{y} = 0$ (the y nullcline); that is, $g(x,y) = 0$. On these curves, direction vectors are horizontal ($\leftarrow$ or $\rightarrow$)

Examples (cont'd from last time)

4) $\begin{cases} \dot{x} = xy - y \\ \dot{y} = xy - x \end{cases}$

death

mutualism

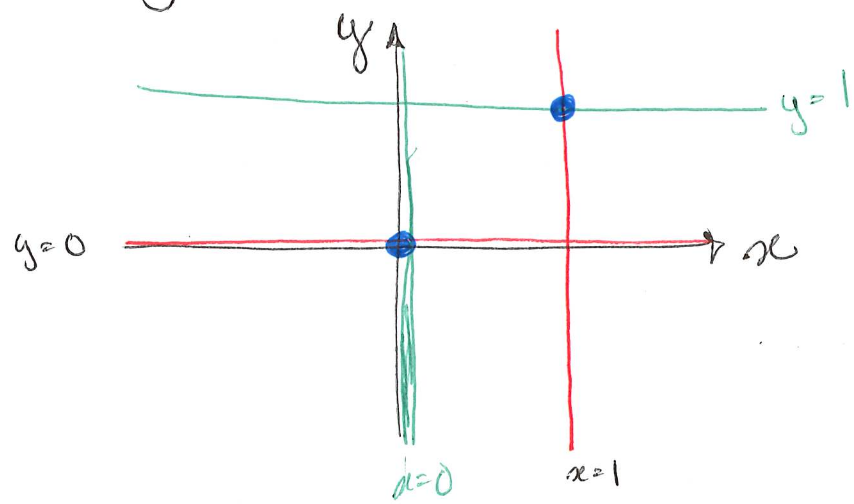
(A possible interpretation of the model is given in green.)

$\begin{cases} f(x,y) = xy - y \\ g(x,y) = xy - x \end{cases}$

$\therefore$ Nullclines are

$f(x,y) = 0 \Leftrightarrow xy - y = 0 \Leftrightarrow y = 0$ or $x = 1$

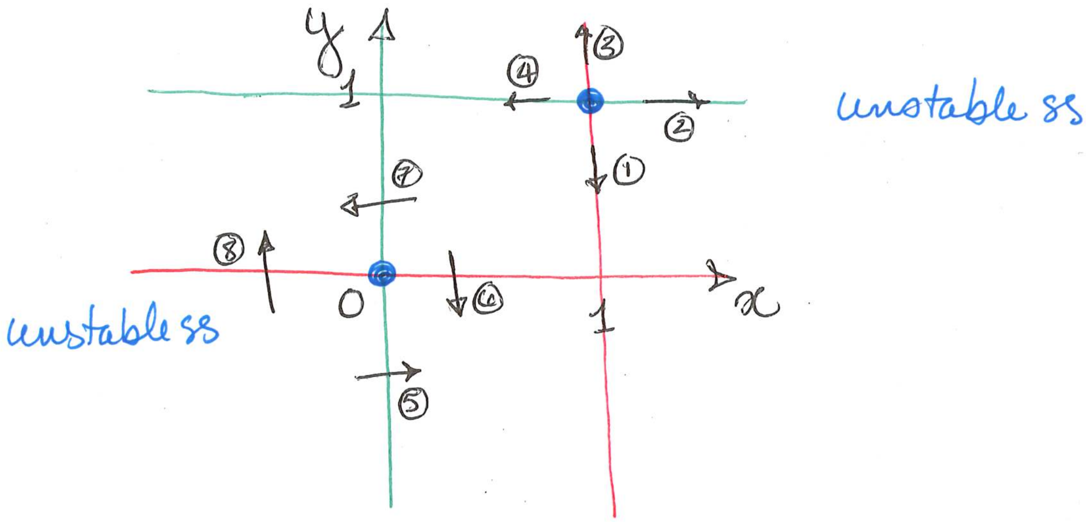
$g(x,y) = 0 \Leftrightarrow xy - x = 0 \Leftrightarrow x = 0$ or $y = 1$



Steady states are $(0,0)$ and $(1,1)$.

Direction field:

- on the x-nullclines, flow is vertical: |
- " " y- " , " " horizontal: —



Find the direction of each tangent vector:

at $(1,1)$

① let $0 < \varepsilon \ll 1$, $\dot{y}(1, 1-\varepsilon) = g(1, 1-\varepsilon) = 1 - \varepsilon - 1 < 0$ ↓

② $\dot{x}(1+\varepsilon, 1) = f(1+\varepsilon, 1) = 1 + \varepsilon - 1 > 0$ →

③ $\dot{y}(1, 1+\varepsilon) = g(1, 1+\varepsilon) = 1 + \varepsilon - 1 > 0$ ↑

④ , $\dot{x}(1-\varepsilon, 1) = f(1-\varepsilon, 1) = 1 - \varepsilon - 1 < 0$ ←

at $(0,0)$

⑤ let $a > 0$, $\dot{x}(0, -a) = f(0, -a) = a > 0$ →

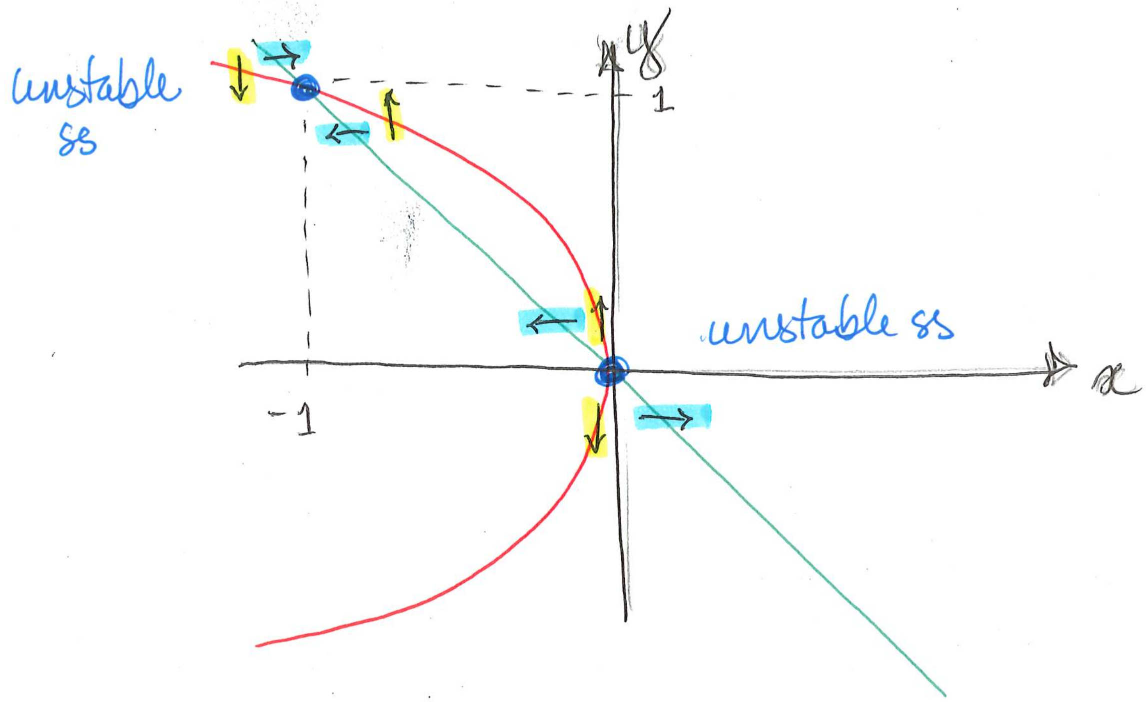
⑥ , $\dot{y}(a, 0) = g(a, 0) = -a < 0$ ↓

⑦ , $\dot{x}(0, a) = f(0, a) = -a < 0$ ←

⑧ , $\dot{y}(-a, 0) = g(-a, 0) = a > 0$ ↑

Go back to examples from last time.

5) $\begin{cases} \dot{x} = x + y^2 \\ \dot{y} = x + y \end{cases}$



Notice that the direction of the $\dot{y}$ vector

- is the same on nullcline segments between steady states
- so direction changes only occur at steady states

The same is true of the $\dot{x}$ vector.

* Dramatic local changes in the flow pattern can only take place in the vicinity of steady states.

Revisiting the exercises from last time...

3) $\begin{cases} \dot{x} = x\sin(y) \\ \dot{y} = x^3 \end{cases}$ Plot the flow-field in plane. We observe:

- For some steady states, the flow moves away (unstable).
- For some steady states, the flow is toward the steady state (stable).
- For some steady states, the flow spirals. (oscillations in $x(t)$ and $y(t)$).

Questions

- When is a steady-state stable or unstable?
- When is there a rotation around the steady-state?

To answer these questions, we zoom into the vicinity of the steady state, close enough so that the system is almost linear.