September 12

Thursday, September 12, 2019 3:29 PM

Homework Hints

$f(X)$ = prob (proportion of the time) that a stubborn individual becomes unstubborn

neighbour $S_{-1}X_{+1}$ focal +1 −1	$\xrightarrow{f(X_{+1})}$ −1	X_{-1} +1	symmetry	$X_{-1}S_{-1} \xrightarrow{f(S_{-1})} S_{-1}$
$S_{-1}X_{+1}$ +1 −1	$\xrightarrow{(1-f(X_{+1}))}$ −1	S_{-1} +1		$X_{-1}S_{-1} \xrightarrow{(1-f(S_{-1}))} X_{-1}$
$X_{-1}X_{-1}$	$\longrightarrow$	X_{-1}		$X_{-1}X_{+1} \longrightarrow X_{+1}$
$S_{-1}S_{+1}$	$\longrightarrow$	S_{-1}		$X_{+1}X_{-1} \longrightarrow X_{-1}$

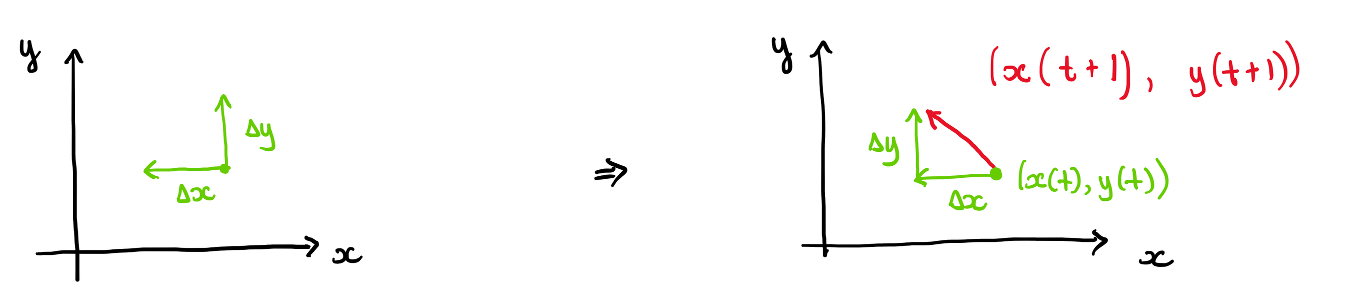
* sometimes switching rate is a function of X, sometimes it's a function of S

Lecture #3

Consider the two-dimensional autonomous ODE system

$$(3)\begin{cases} \dot{x} = \dfrac{dx}{dt} = f(x,y) \\ \dot{y} = \dfrac{dy}{dt} = g(x,y) \end{cases} \quad \text{or} \quad \begin{rcases} \dfrac{\Delta x}{\Delta t} = f(x,y) \\ \dfrac{\Delta y}{\Delta t} = g(x,y) \end{rcases} (4)$$

Last time, we observed that we can evaluate $f(x,y)$ & $g(x,y)$ at any point (x,y) & draw an arrow representing the movement of the solution in the phase plane, over a small interval Δt (eq. 4)



For small enough Δt & repeating this procedure, we can trace the solution $\underbrace{(x(t),\ y(t))}_{\text{vector function}}$ in the phase plane

Summary of Facts about Vector Functions (from Calculus)

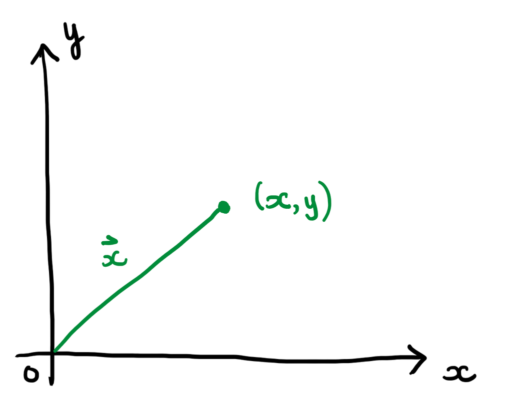
1. The pair $(x(t),\ y(t))$ represents a curve in the (x,y) plane w/ t as a parameter

2. $\vec{x}(t) = (x(t),\ y(t))$ also represents a position vector: a vector w/ tail at origin $(0,0)$ & head at $(x(t),\ y(t))$

3. The vector $\dfrac{d\vec{x}}{dt} = \dot{\vec{x}} = \left(\dfrac{dx}{dt}, \dfrac{dy}{dt}\right) = (\dot{x}, \dot{y})$ has a well-defined geometric meaning. It is a vector that is tangent to a solution curve at $\vec{x}(t)$. Its magnitude, $|\dot{x}(t)|$ represents the speed of motion of the point $\vec{x}(t)$ along the curve.

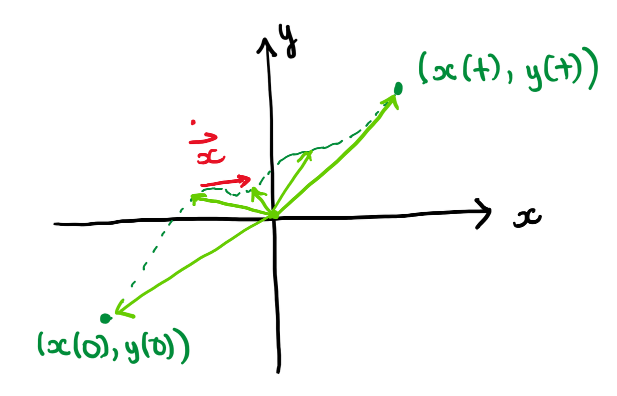
4. The set of eq. 3 can be written in vector form
$\dot{\vec{x}} = \vec{F}(\vec{x})$ .... (5)

   The vector function $\vec{F} = (f,g)$, assigns a vector to every location $\vec{x}$ in the plane, where $\vec{x}$ is the position vector & $\dot{\vec{x}}$ is the velocity vector.



. (x,y) is a point on the solution curve

. $\vec{x}$ is a vector that points to (x,y)



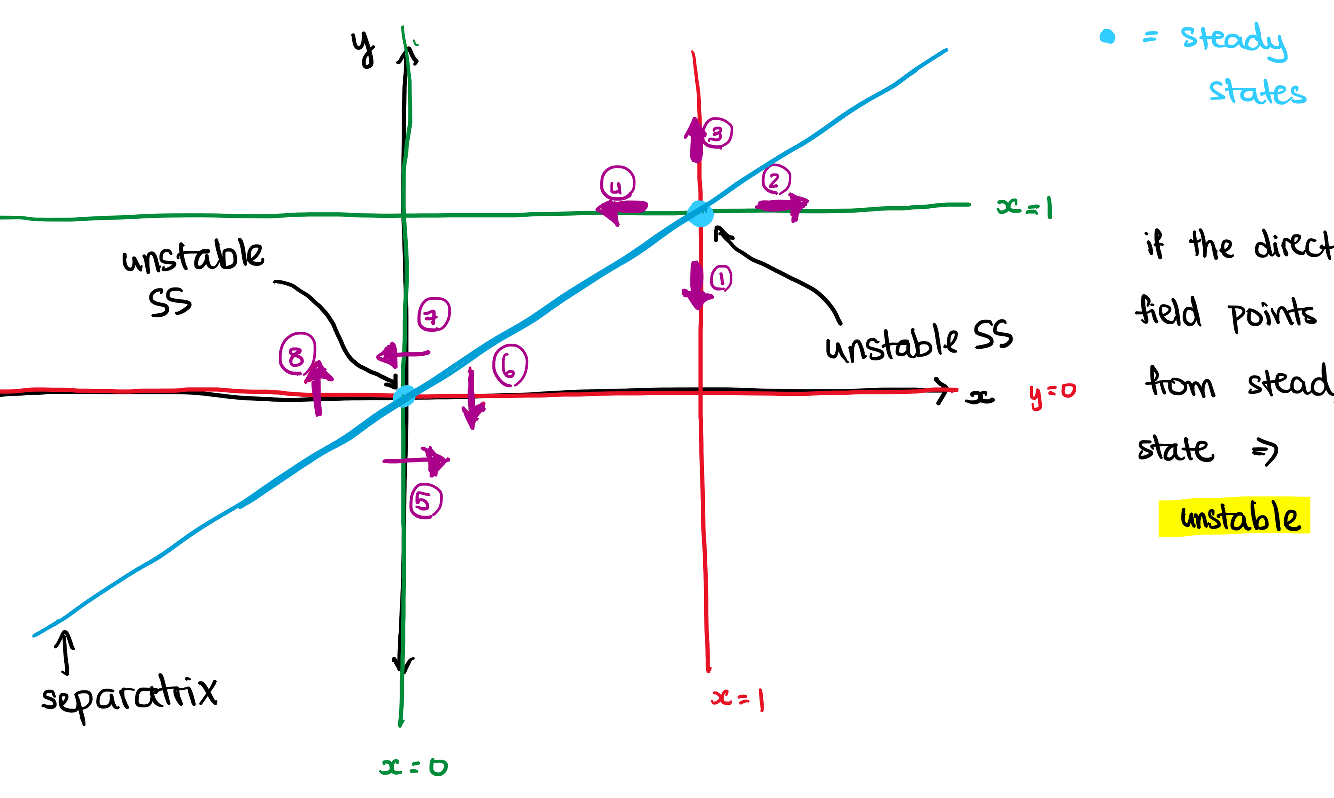
Examples

4) $\dot{x} = xy - y$ $\quad\therefore\quad$ $f(x,y) = xy - y$
$\dot{y} = xy - x$ $\qquad\quad$ $g(x,y) = xy - x$

Find Nullclines

$f(x,y) = 0 \iff xy - y = 0 \iff y = 0$ or $x = 1$

$g(x,y) = 0 \iff xy - x = 0 \iff x = 0$ or $y = 1$



if the direction field points away from steady state ⇒ unstable

* notice how at (1,0), it's not a steady state
↳ the $f(x,y)$ & $g(x,y)$ must cross for it to be a steady state.

Direction Field

- On the x-nullclines, $\dot{x} = 0$ ⇒ flow is vertical
- " y-nullclines, $\dot{y} = 0$ ⇒ " horizontal

Find the direction of each tangent vector ⓘ:

At (1,1) let $0 < \varepsilon \ll 1$

① vertical flow given by $\dot{y}$ or g
$g(1, 1-\varepsilon) = (1-\varepsilon) - 1 = -\varepsilon < 0$ ↓

② horizontal flow given by $\dot{x}$ or f
$f(1+\varepsilon, 1) = (1+\varepsilon)\cdot 1 - 1 = \varepsilon > 0$ →

③ $g(1, 1+\varepsilon) = (1+\varepsilon) - 1 = \varepsilon > 0$ ↑

④ $f(1-\varepsilon, 1) = (1-\varepsilon)\cdot 1 - 1 = -\varepsilon < 0$ ←

⑤ $f(0, -\varepsilon) = --\varepsilon = \varepsilon > 0$ →

⑥ $g(\varepsilon, 0) = -\varepsilon = -\varepsilon < 0$ ↓

⑦ $f(0, \varepsilon) = -\varepsilon = -\varepsilon < 0$ ←

⑧ $f(-\varepsilon, 0) = --\varepsilon = \varepsilon > 0$ ↑

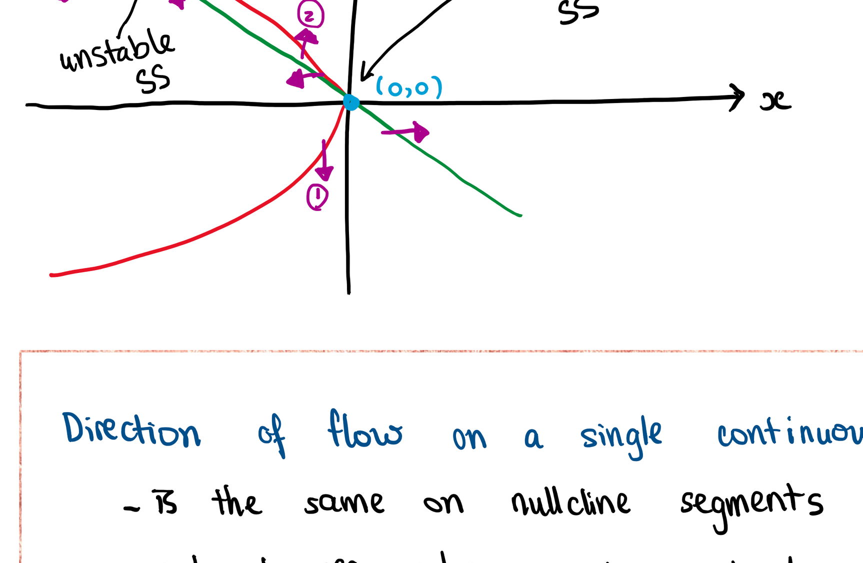
☒

Separatrix
- divides the plane into regions where the solution outcomes are very different.
- join steady states

* Flows are drastically different near the steady states
↳ but are rather uniform otherwise.

Examples

5) $\dot{x} = x + y^2 = f(x,y) \iff x = -y^2$
$\dot{y} = x + y = g(x,y) \iff y = -x$



① $g(-\varepsilon, -\varepsilon) = -\varepsilon - \varepsilon = -2\varepsilon < 0$ ↓
② $g(-\varepsilon, 2\varepsilon) = \varepsilon > 0$ ↑
③ $g(-1+\varepsilon,\ 1-\frac{1}{2}\varepsilon) = \frac{1}{2}\varepsilon > 0$ ↑
④ $g(-1-\varepsilon,\ 1+\frac{1}{2}\varepsilon) = -\frac{1}{2}\varepsilon < 0$ ↓

☒

Direction of flow on a single continuous nullcline
- is the same on nullcline segments between steady states
- only changes when crossing steady states

Catalogue of SS Behaviours

- **Unstable**: flow moves away from steady state
- **Stable**: " towards into "
- **Oscillation**: flow spirals