Lecture#4

Dramatic changes in the flowfield only take place...

Close to the Steady States

Suppose we have the system

$$\begin{cases} \dot{X} = F(X,Y), \\ \dot{Y} = G(X,Y), \end{cases} \quad \cdots\cdots\cdots (5)$$

where F and G are nonlinear functions. Assume that (X^*, Y^*) is a steady state, i.e.

$$F(X^*, Y^*) = G(X^*, Y^*) = 0.$$

Now consider the close-to-steady-state solutions

$$\begin{cases} X(t) = X^* + x(t), \\ Y(t) = Y^* + y(t), \end{cases} \quad \cdots\cdots (6)$$

where $0 < x(t) \ll 1$, and $0 < y(t) \ll 1$. The functions $x(t)$ and $y(t)$ are perturbations of the steady state.

↳ perturbation theory!

Substituting (6) into (5), we obtain

$$\begin{cases} \dfrac{d}{dt}(X^* + x) = F(X^*+x, Y^*+y) \\ \dfrac{d}{dt}(Y^*+y) = G(X^*+x, Y^*+y) \end{cases} \quad \ldots\ldots (7)$$

Begin with (7a):

$$LHS = \frac{d}{dt}(X^*+x) = \underbrace{\cancel{\frac{dX^*}{dt}}}_{\to 0} + \frac{dx}{dt} = \dot{x}$$

$$RHS = F(X^*+x, Y^*+y)$$ do a Taylor series expansion ◡̈

$$= \underbrace{\cancel{F(X^*, Y^*)}}_{\to 0} + \left.\frac{\partial F}{\partial X}\right|_{(X^*,Y^*)} x + \left.\frac{\partial F}{\partial Y}\right|_{(X^*,Y^*)} y + h.o.t.$$

we only want LINEAR terms

where h.o.t. = higher order terms (terms of order x^2, y^2, xy, and higher)

Thus, (7a) becomes, to first order:

$$\dot{x} = F_X\big|_* x + F_Y\big|_* y \quad \ldots\ldots (8a)$$

Similarly, (7b) becomes, to first order:

$$\dot{y} = G_x\Big|_* x + G_y\Big|_* y \quad \ldots\ldots \quad (8b)$$

Putting (8a) + (8b) together, we arrive at the linearisation of (5) about the steady state (X^*, Y^*):

$$\begin{cases} \dot{x} = F_x\Big|_* x + F_y\Big|_* y \\ \dot{y} = G_x\Big|_* x + G_y\Big|_* y \end{cases} \quad \ldots\ldots \quad (8)$$

Thus, we can write

$$\boxed{\frac{d\vec{x}}{dt} = A\vec{x}}, \quad \text{where } A = \begin{bmatrix} F_x\Big|_* & F_y\Big|_* \\ G_x\Big|_* & G_y\Big|_* \end{bmatrix} \quad \ldots (9)$$

(9) is a linear system of ODEs. How do we solve it?

<u>Brief Review</u>

i) First Order

$$\frac{dx}{dt} = Kx \quad \ldots (10) \quad \rightsquigarrow \quad x(t) = x_0 e^{Kt} \quad \ldots (11)$$

(using separation of variables)

where $x_0 = x(0)$.

(ii) Second Order

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c = 0 \quad \ldots\ldots \quad (12)$$

If we assume that solutions are of the form $x = e^{\lambda t}$ (inspired by (11)), and plug this form into (12), we obtain the characteristic equation

$$a\lambda^2 + b\lambda + c = 0 \iff \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \ldots (13)$$

So there are two eigenvalues (either distinct or not). If the eigenvalues are distinct, we have

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \quad \ldots\ldots \quad (14)$$

If $\lambda_1 = \lambda_2$, then we have

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} \quad \ldots\ldots \quad (15).$$

If the eigenvalues are complex then (14) still holds, but we write $\lambda = \underline{a} \pm ib$, and obtain the real-valued solutions

$$x(t) = e^{at}\left(c_1 \cos(bt) + c_2 \sin(bt)\right) \ldots (16).$$

iii) Order n

$$\dot{\vec{x}} = A\vec{x} \qquad \ldots\ldots\ldots\ldots \quad (17)$$

where A is an $n \times n$ matrix and $\vec{x}$ is an n-vector. Inspired by cases (i) + (ii), we assume solutions of the form

$$\vec{x} = \vec{v}e^{\lambda t} \qquad \ldots\ldots\ldots\ldots \quad (18)$$

where $\vec{v}$ is a constant n-vector. We then ~~and~~ substitute (18) into (17). We obtain

from (17)

$$\dot{\vec{x}} = \lambda\vec{v}e^{\lambda t} = A\vec{v}e^{\lambda t} \qquad \ldots\ldots \quad (19)$$

from differentiating (18)

Cancelling the common factor of $e^{\lambda t}$ we obtain

$$A\vec{v} = \lambda\vec{v} \iff \boxed{(A - \lambda I)\vec{v} = \vec{0}} \qquad \ldots \quad (20)$$

fundamental eqn in linear algebra! The λ's must be eigenvalues of A!

So there must be n eigenvalues (distinct or not), + solutions must be linear combinations of the independent eigenfunctions $e^{\lambda_i t}$.

As in (ii), complex eigenvalues lead to oscillatory eigenfunctions, non-distinct eigenvalues lead to eigenfunctions of the form $t^q e^{\lambda t}$, $q = 0, 1, \ldots, m-1$ where m is the multiplicity of the eigenvalue λ.

Examples

1. Solve

$$\begin{cases} \dot{x} = 3x - y \\ \dot{y} = 6x - 4y \end{cases} \iff \dot{\vec{x}} = \begin{bmatrix} 3 & -1 \\ 6 & -4 \end{bmatrix} \vec{x}$$

eigenvalues

$A = \begin{bmatrix} 3 & -1 \\ 6 & -4 \end{bmatrix}$ & the eigenvalues are given by

$$\begin{vmatrix} 3-\lambda & -1 \\ 6 & -4-\lambda \end{vmatrix} = 0 \iff (3-\lambda)(-4-\lambda) + 6 = 0 \iff /$$

$$/ \iff \lambda^2 + \lambda - 12 + 6 = 0 \iff \lambda^2 + \lambda - 6 = 0$$

$$\iff (\lambda+3)(\lambda-2) = 0,$$ two distinct, real eigenvalues

eigenvectors

$\lambda_1 = 2$:

$$(A - \lambda I)\vec{v}_1 = \vec{0} \iff \begin{bmatrix} 3-2 & -1 \\ 6 & -4-2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff //$$

$$// \iff \begin{bmatrix} 1 & -1 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

one row is redundant

$\therefore v_{11} - v_{12} = 0$, & we can choose

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = -3:}$

$$(A - \lambda I)\vec{v}_2 = \vec{0} \Leftrightarrow \begin{bmatrix} 3-(-3) & -1 \\ 6 & -4-(-3) \end{bmatrix} \vec{v}_2 = \vec{0} \Leftrightarrow (*)$$

$$(*) \Leftrightarrow \begin{bmatrix} 6 & -1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

one row is redundant

$\therefore 6v_{21} - v_{22} = 0$, & we can choose

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Solutions:

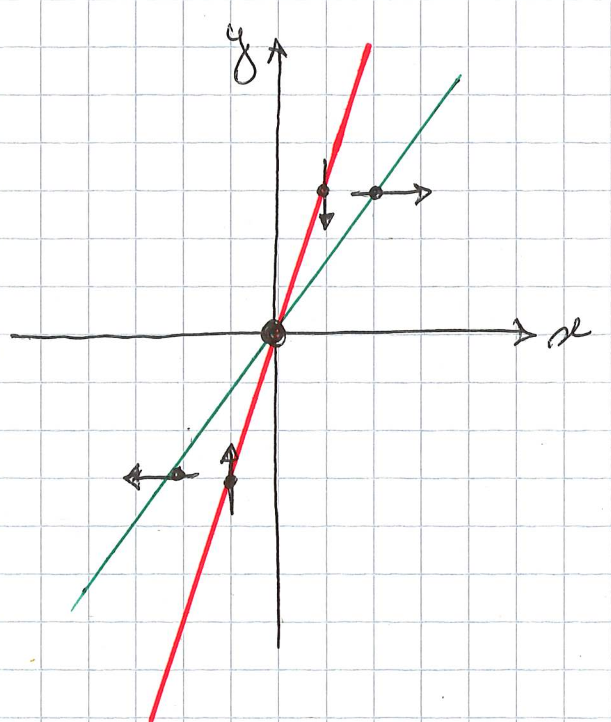
$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{-3t}$$

Let's relate this back to the phase plane.

$$\begin{cases} \dot{x} = 3x - y & = f(x,y) \\ \dot{y} = 6x - 4y & = g(x,y) \end{cases}$$

nullclines:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \text{ yes } \begin{cases} 3x - y = 0 \\ 6x - 4y = 0 \end{cases} \text{ yes } \begin{cases} y = 3x \\ y = \frac{3}{2}x \end{cases}$$



$\therefore$ the steady-state is at the origin & the nullclines are straight lines through the origin.

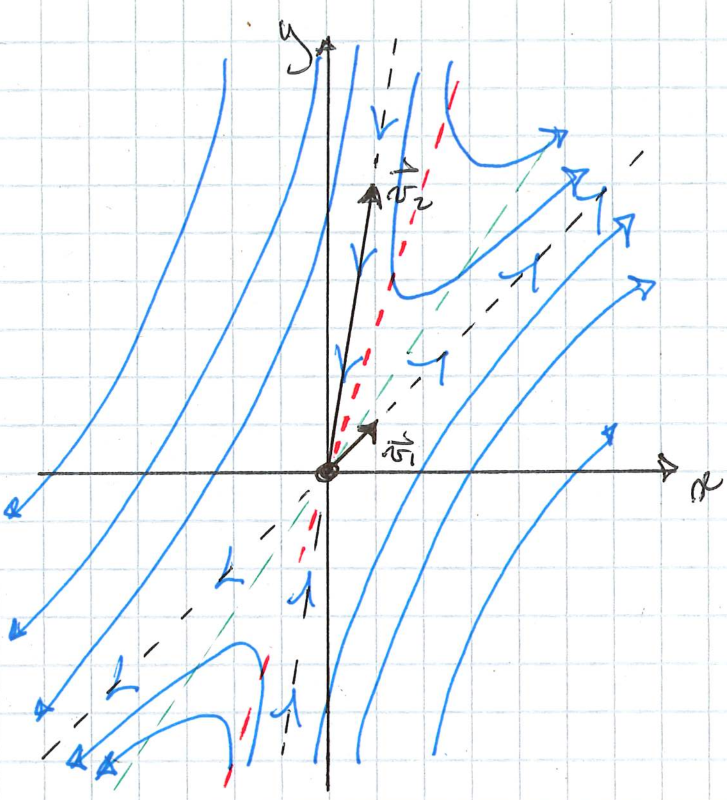
We determine flow directions as before:

$$g(1,3) = 6 - 12 = -6 \downarrow$$
$$g(-1,-3) = -6 + 12 = 6 \uparrow$$

$$f(2,3) = 6 - 3 = 3 \rightarrow$$
$$f(-2,-3) = -6 + 3 = -3 \leftarrow$$

So flow is toward the steady state across the x-nullcline
& " " away from " " " " " y- " .

How does this relate to the solution $\vec{x}(t)$? We plot the eigenvectors:



Notice that

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{-3t}$$

This component gets bigger as t increases ⇒ flow moves away from (0,0)

This component gets smaller as t increases ⇒ flow moves towards (0,0)

So the nullclines are useful for figuring out flow directions in the plane, and these flow directions are consistent with the eigenvector & eigenvalue solutions.

Notice that $\vec{v}_1$ & $\vec{v}_2$ define the separatrices of the steady state. Once on a separatrix, the flow stays on the separatrix.

Examples cont'd:

2. Solve

$\dot{\vec{x}} = A\vec{x}$ where $A =$

Ex 2 taken fr Nagle, Saff, + Snyder, 5th Ed, Chap 9.6, exercise 7. fundamental matrix is

$$\begin{bmatrix} 1 & \cos(t) & \sin(t) \\ 0 & -\cos(t) & -\sin(t) \\ 0 & -\sin(t) & \cos(t) \end{bmatrix}$$