# September 17

Tuesday, September 17, 2019 3:28 PM

- Dramatic changes in the flow field only occurs close to the steady state

Suppose

$$\begin{cases} \dot{X} = F(X, Y) \\ \dot{Y} = G(X, Y) \end{cases} \quad \dots\dots (5)$$

where $F$ & $G$ are non-linear functions

Assume that $(X^*, Y^*)$ is a steady state of (5)

$$\begin{cases} F(X^*, Y^*) = 0 \\ G(X^*, Y^*) = 0 \end{cases}$$

Now consider the close-to-steady-state solutions

$$\begin{cases} X(t) = X^* + x(t) \\ Y(t) = Y^* + y(t) \end{cases} \quad \dots\dots (6)$$

where $0 < x(t) \ll 1$ and $0 < y(t) \ll 1$.

. The functions $x(t)$ & $y(t)$ are **perturbations** of $(X^*, Y^*)$

↳ perturbation theory

↳ asymptotic analysis

Plug (6) into (5)

$$\begin{cases} \dot{x} = F(X^* + x, Y^* + y) \\ \dot{y} = G(X^* + x, Y^* + y) \end{cases} \quad \dots\dots (7)$$

Consider the RHS of (7a)

$$RHS = F(X^* + x, Y^* + y) \qquad \text{do a Taylor Series Expansion}$$

$$= \underbrace{F(X^*, Y^*)}_{0^{th}\text{ order}} + \underbrace{\frac{\partial F}{\partial x}\Big|_* x + \frac{\partial F}{\partial y}\Big|_* y}_{1^{st}\text{ order}} + \text{higher order terms (h.o.t)}$$

↳ 2nd order + 3rd order (too small)

⇒ ignore

∴ (7a) to first order is

$$\begin{cases} \dot{x} = \frac{\partial F}{\partial x}\Big|_* x + \frac{\partial F}{\partial y}\Big|_* y \\ \dot{y} = \frac{\partial G}{\partial x}\Big|_* x + \frac{\partial G}{\partial y}\Big|_* y \end{cases} \quad \dots\dots (8)$$

Write

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \text{where } A = \begin{bmatrix} F_X^* & F_y^* \\ G_X^* & G_y^* \end{bmatrix} \quad \dots\dots (9)$$

(9) is a linear system of ODEs.

## Brief Review: Solving linear ODEs

### i) First Order

$$\frac{dx}{dt} = kx \quad \dots (10)$$

Solutions: $x(t) = x_0 e^{kt}$ .... (11)

### ii) Second Order

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c = 0 \quad \dots\dots (12) \qquad a \neq 0$$

If we assume that $x(t) \propto e^{rt}$, then we obtain

$$ar^2 + br + c = 0 \quad \dots (13)$$

← characteristic eq.

$$\Leftrightarrow r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \dots (14)$$

If $r_1 \neq r_2$, then

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad \dots (15)$$

If $r_1 = r_2 = r$ then

$$x(t) = c_1 e^{rt} + c_2 t e^{rt} \quad \dots (16)$$

If $r_1 \neq r_2$, complex

$$r = a \pm ib$$

$$x(t) = e^{at}(c_1 \cos(bt) + c_2 \sin(bt)) \quad \dots (17)$$

### iii) Order n

$\dot{x} = A\vec{x}$ where $A$ is an $n \times n$ matrix & $X$ is an n-vector

. We assume

$$\vec{x} = \vec{v} e^{\lambda t} \quad \dots (18)$$

($\lambda$ ← eigenvalue; $\vec{v}$ ↳ constant (eigenvector))

Plug (18) into (17)

$$\dot{\vec{x}} = \lambda \vec{v} e^{\lambda t} = A\vec{x}e^{\lambda t} \quad \dots\dots (19)$$

Cancel $e^{\lambda t}$ + rerrange

$$A\vec{v} - \lambda\vec{v} = 0 \Leftrightarrow \boxed{(A - \lambda I)\vec{v} = 0} \quad \dots (20)$$

In order for (20) to have nontrivial solutions

$$(A - \lambda I) = 0$$

↳ That will be an eigvalues (distinct or not) & solutions are linear combinations of the eigenfunctions $e^{\lambda_i t}$.

As in (ii)

- complex eigenvalues ⇒ oscillatory sol'n
- duplicate eigenvalues ⇒ $te^{\lambda t}, t^2e^{\lambda t}, \dots t^{n-1}e^{\lambda t}$

where $n$ is the multiplicity of the eigenvalue.

## Example 1

1. Solve

$$\begin{cases} \dot{x} = 3x - y \\ \dot{y} = 6x - 4y \end{cases} \Leftrightarrow \dot{\vec{x}} = A\vec{x} \text{ where } A = \begin{bmatrix} 3 & -1 \\ 6 & -4 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 3-\lambda & -1 \\ 6 & -4-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (3-\lambda)(-4-\lambda) + 6 = 0$$

$$\Leftrightarrow \lambda^2 + \lambda - 6 = 0$$

$$\Leftrightarrow (\lambda + 3)(\lambda - 2) = 0$$

$$\lambda_2 = -3 \qquad \lambda_1 = 2$$

Eigenvectors (use (20))

$\lambda_1 = 2$

$$\begin{bmatrix} 3-2 & -1 \\ 6 & -4-2 \end{bmatrix}\vec{v} = \vec{0}$$

$$\Leftrightarrow \begin{bmatrix} 1 & -1 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} \vec{v}_{11} \\ \vec{v}_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

one row is redundant

Solve

$$v_{11} - v_{12} = 0 \Leftrightarrow v_{11} = v_{12}$$

to make life easy, choose $v_{12} = 1$

$$\therefore \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -3$ $\quad (A - \lambda I)\vec{v}_2 = \vec{0}$

* good to check that one row is redundant.

$$\begin{bmatrix} 6 & -1 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

one row is redundant

$$\therefore 6v_{21} - v_{22} = 0$$

Choose $v_{21} = 1$, $\therefore v_{22} = 6$

$$\therefore \vec{v}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

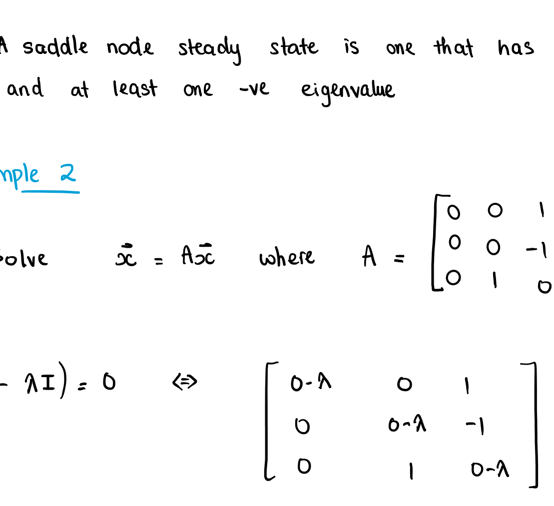
Solutions are

$$\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 6 \end{bmatrix}e^{3t} \quad ☒$$

## Phase Plane

### Nullclines

$$\begin{cases} \dot{X} = 0 \\ \dot{Y} = 0 \end{cases} \Leftrightarrow \begin{cases} y = 3x \\ y = \frac{3}{2}x \end{cases}$$




- a steady state like this is called a **saddle node**

. A saddle node steady state is one that has at least one +ve eigenvalue and at least one -ve eigenvalue

## Example 2

Solve $\dot{\vec{x}} = A\vec{x}$ where $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$$(A - \lambda I) = 0 \Leftrightarrow \begin{bmatrix} 0-\lambda & 0 & 1 \\ 0 & 0-\lambda & -1 \\ 0 & 1 & 0-\lambda \end{bmatrix} = 0$$

$$\Leftrightarrow -\lambda[(-\lambda)(-\lambda) - (-1)(1)] = 0$$

$$\Leftrightarrow -\lambda[\lambda^2 + 1] = 0$$

$$\boxed{\lambda_1 = 0, \ \lambda_2 = i, \ \lambda_3 = -i} \qquad \lambda = \frac{\pm\sqrt{-4}}{2} = \frac{\pm\sqrt{-4}}{2}$$

Eigenvectors - maple

| | | |
|---|---|---|
| $\lambda_1 = 0$ | $\lambda_2 = i$ | $\lambda_3 = -i$ |
| $v_1 =$ | $v_2 =$ | $v_3 =$ |

(use Maple to find vectors)

Solution

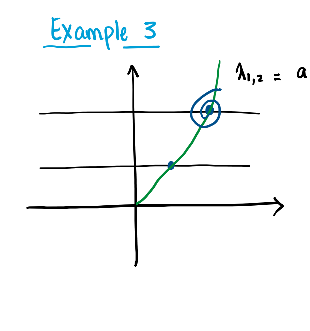
$$\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} \cos(t) \\ -\cos(t) \\ -\sin(t) \end{bmatrix} + c_3\begin{bmatrix} \sin(t) \\ -\sin(t) \\ \cos(t) \end{bmatrix}$$

- periodic solutions

↳ neither increasing or decreasing

∴ steady state is a centre

## Example 3



$\lambda_{1,2} = a + ib_1$ $\quad a < 0$, $b \neq 0$