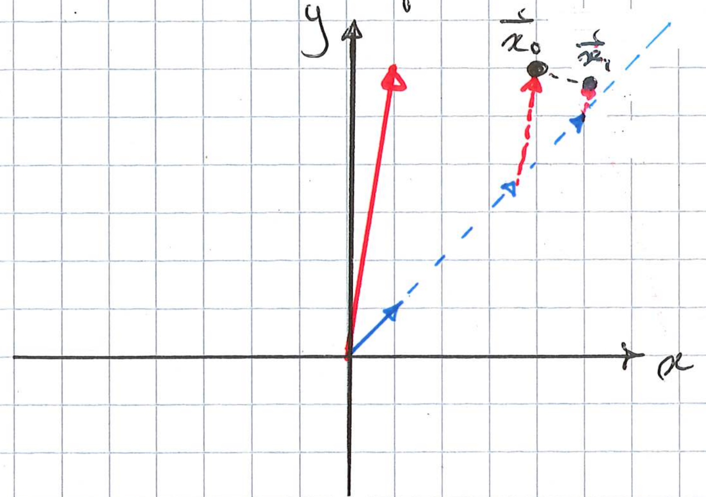
Lecture #5

Flow from Eigenvalues & Eigenvectors

Revisit Ex 1 from last time:



$$\vec{x}(t) = c_1 \underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\vec{v}_1} e^{2t} + c_2 \underbrace{\begin{bmatrix}1\\6\end{bmatrix}}_{\vec{v}_2} e^{-3t}$$

$= c_1\vec{v}_1$

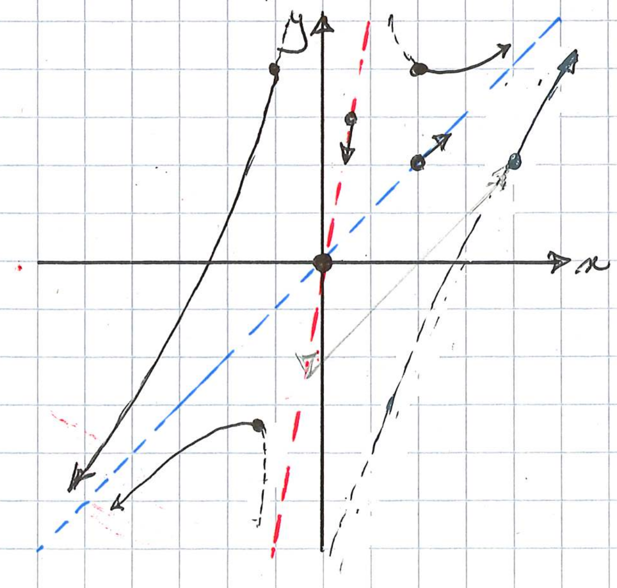
$= c_2\vec{v}_2$

Now, let t increase.

$$\begin{cases} c_1\vec{v}_1 e^{2t} > c_1\vec{v}_1 \\ c_2\vec{v}_2 e^{-3t} < c_2\vec{v}_2 \end{cases} \Rightarrow \vec{x}_1$$

As $t \to \infty$, we expect $\vec{x}(t)$ to approach the line defined by $\vec{v}_1$.

Repeat for other starting points, and running time both backwards & forwards



Eigenvectors ⇝ define the separatrices

The separatrices are invariant sets

Proof that the line $y = x$ is an invariant set ... (next p)

Define $L = x - y$. The line $L = 0$ is an invariant set for the dynamical system

$$\begin{cases} \dot{x} = 3x - y \\ \dot{y} = 6x - 4y \end{cases}$$

if

$$\dot{L} = 0 \Leftrightarrow \dot{x} - \dot{y} = 0 \Leftrightarrow 3x - y = 6x - 4y$$

$$\Leftrightarrow 3x - 3y = 0 \Leftrightarrow x = y$$

Thus, the separatrix defined by $\vec{v}_1$ is an invariant set for the dynamical system. ▨

Note: L is an $\boxed{\text{unstable manifold}}$ for the dynamical system.

When is a Steady State Stable?

From lecture #4, solutions of $\dot{\vec{x}} = A\vec{x}$ are

$$\vec{x}(t) = \sum_{i=1}^{n} c_i \vec{v}_i e^{\lambda_i t} \quad \ldots\ldots\ldots (21)$$

where

$\boxed{\text{case 1: all } \lambda_i \in \mathbb{R}}$

c_i = constants determined by the initial conditions
$\vec{v}_i$ = eigenvectors of A
λ_i = eigenvalues of A
n = length of $\vec{x}$ + dimension of ea side of A

(Note: We show here the case of n distinct eigenvectors + eigenvalues. If there are any eigenvalues of multiplicity larger than 1, the solution (21) is a little different, but the principles remain the same.)

$$\boxed{\begin{array}{llll} \text{Solutions } \vec{x}(t) \text{ grow if any of the } \lambda_i > 0 & \rightsquigarrow \text{unstable} \\ \qquad\qquad\quad \text{decay " all " " } \lambda_i < 0 & \rightsquigarrow \text{stable} \end{array}}$$

$\boxed{\text{case 2 : } m \text{ pairs of complex } \lambda_i}$ Then

$$\lambda_i = a_i + i b_i$$

and

$$\vec{x}(t) = \sum_{i=1}^{m} e^{a_i t}\left(c_{1i}\cos(b_i t) + c_{2i}\sin(b_i t)\right) + \sum_{i=2m+1}^{n} c_i \vec{v}_i e^{\lambda_i t} \quad \ldots (22)$$

$\boxed{\begin{array}{l}\text{Solutions } \vec{x}(t) \text{ grow if any of the } \lambda_i \text{ or } a_i > 0 \rightsquigarrow \text{unstable} \\ \text{decay " all " " } \lambda_i \;\&\; a_i < 0 \rightsquigarrow \text{stable}\end{array}}$

In 2-D:

$$\begin{cases} \dot{x} = a_{11} x + a_{12} y \\ \dot{y} = a_{21} x + a_{22} y \end{cases} \quad \therefore A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\therefore |A - \lambda I| = 0 \Leftrightarrow (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

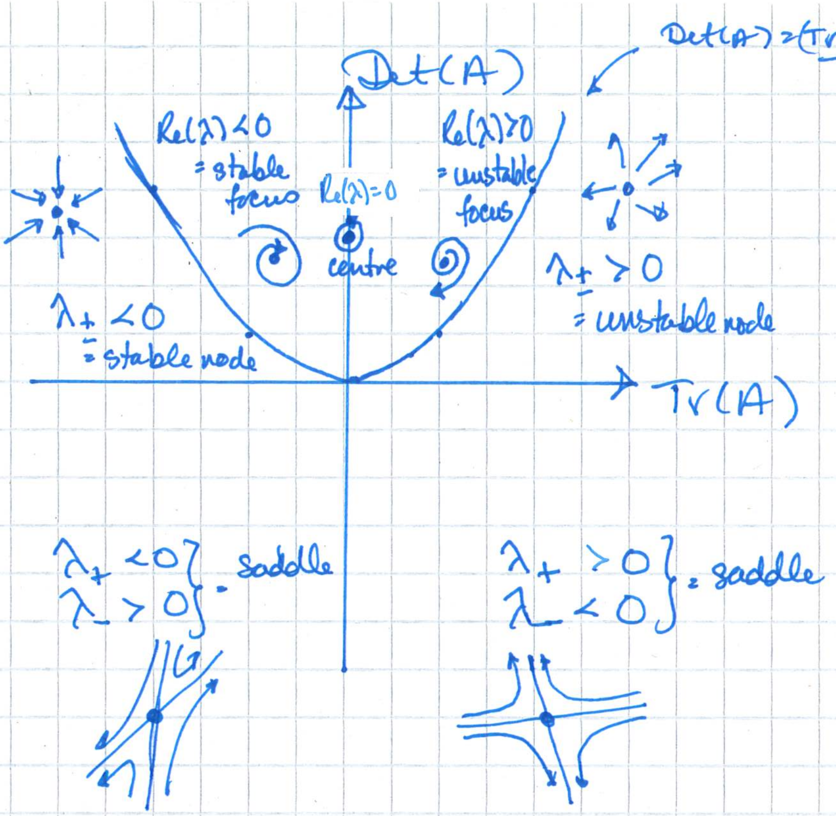
$$\Leftrightarrow \lambda^2 - \underbrace{(a_{11} + a_{22})}_{\boxed{Tr(A)}} \lambda + \underbrace{(a_{11}a_{22} - a_{21}a_{12})}_{\boxed{Det(A)}} = 0$$

$$\Leftrightarrow \lambda = \frac{-Tr(A) \pm \sqrt{(Tr(A))^2 - 4Det(A)}}{2}$$

Let

$$\lambda_+ = \left[-Tr(A) + \sqrt{(Tr(A))^2 - 4Det(A)}\right]\frac{1}{2}$$

$$\lambda_- = \left[-Tr(A) - \sqrt{(Tr(A))^2 - 4Det(A)}\right]\frac{1}{2}$$



$\lambda \in \mathbb{R}$ if

$$(Tr(A))^2 - 4Det(A) \geq 0 \iff$$

$$\iff Det(A) \leq \frac{(Tr(A))^2}{4}$$

Only saddle nodes have separatrices + linear invariant sets.

Example

1. $\dot{\vec{x}} = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix} \vec{x}$ $\quad \lambda = \pm 2i$, $\therefore$ (0,0) is a centre

2. $\dot{\vec{x}} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \vec{x}$ $\quad \lambda_1 = 3$, $\lambda_2 = 2$, $\therefore$ (0,0) is an unstable node

~~3. (from a. (a) Use pplane to find the eigenvalues of the~~ steady states of the dynamical system. Determine the stability.

$$\begin{cases} \dot{x} = 10xy - 1 \\ \dot{y} = x + y - 1 \end{cases}$$

(b) What do you observe about the stable + unstable manifolds?

Ans:

a) The upper steady state is an unstable node. The lower steady state is a saddle node.

b) The upper + lower steady states are connected by the stable manifold of the saddle node.

~~(see fig next p).~~