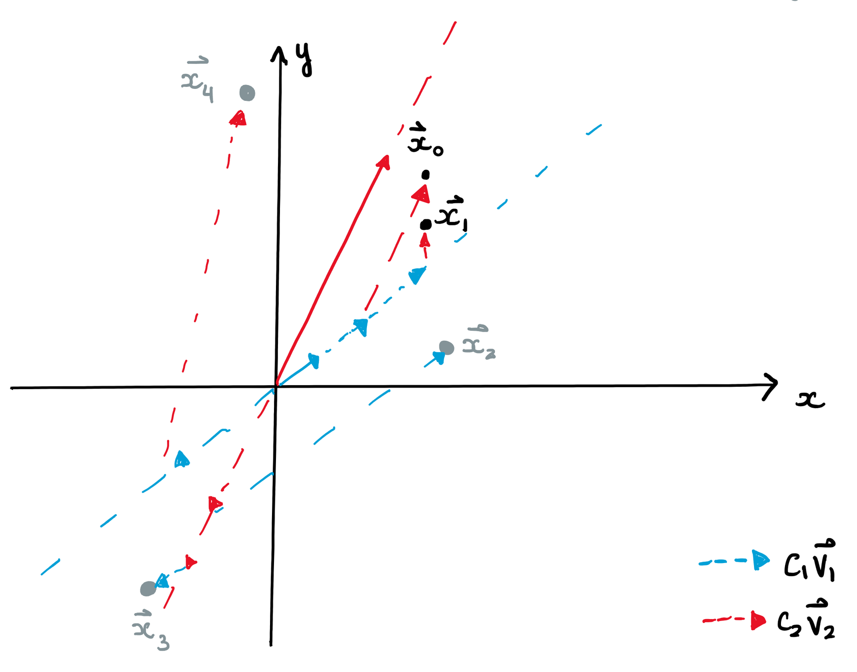
Lecture 5

Thursday, September 19, 2019 3:25 PM

Flow from eigenvalues + eigenvectors

Revisit example 1 from last time

* eigenvalues are important
* eigenvectors are important for only one SS



$$\vec{x}(t) = c_1\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\vec{v}_1}e^{2t} + c_2\underbrace{\begin{bmatrix}1\\6\end{bmatrix}}_{\vec{v}_2}e^{-3t}$$

* $\vec{x}_0$ is a starting point
↳ gives initial conditions
↳ gives c_1 and c_2

Now, let t increase:

$c_1\vec{v}_1e^{2t} > c_1\vec{v}_1$ getting longer

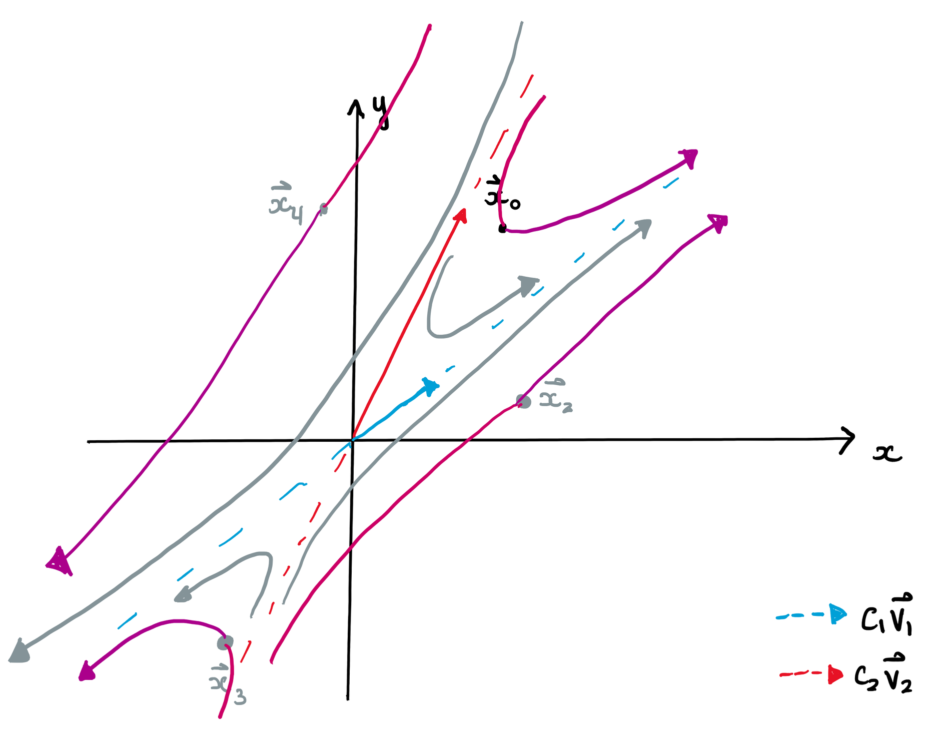
$c_2\vec{v}_2e^{-3t} < c_2\vec{v}_2$ getting shorter

What happened?

$\vec{x}_0 \to \vec{x}_1$ which is closer to the line defined by $\vec{v}_1$

We can see that as $t \to \infty$, $\vec{x}(t) \to$ the line defined by $\vec{v}_1$

$\vec{x}_2$ as $t \to \infty$, $\vec{x}(t) \to$ the line defined by $\vec{v}_1$
$\vec{x}_3$ " " $\vec{v}_1$
$\vec{x}_4$ " " $\vec{v}_1$



- all $\vec{x}_0$ points (initial) lead to trajectories that end upon the line defined by $\vec{v}_1$
- if we run time backwards $(t \to \infty)$, then all starting points $\vec{x}_0$ end up asymptotically close to the line defined by $\vec{v}_2$

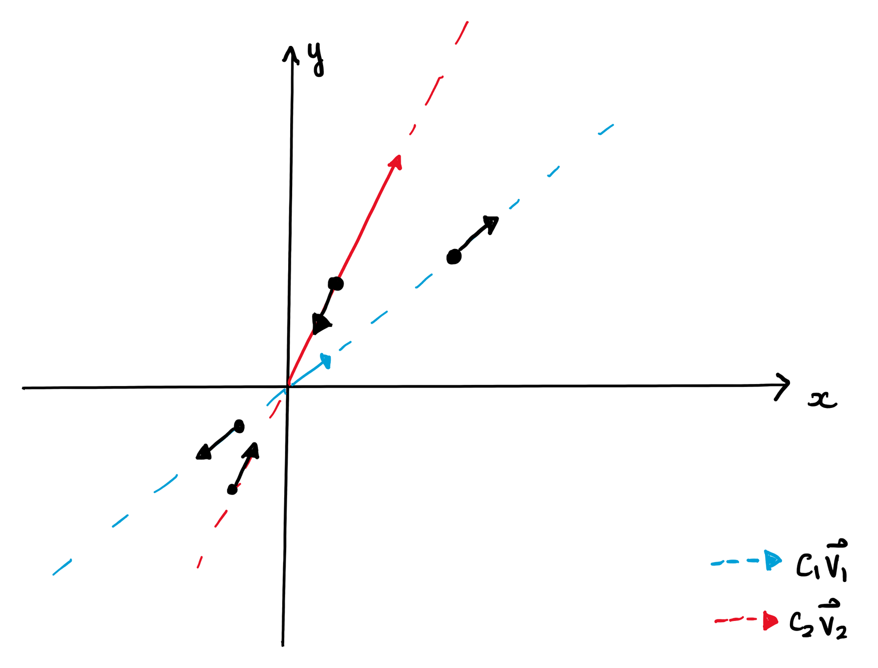
The eigenvectors define the separatrices.
↳ defines basins of attraction

These separatrices are only generated for systems where $\lambda_1 > 0$, $\lambda_2 > 0$.

On the case of our example:

i) the red separatrix defines the two basins of attraction for $t \to +\infty$ in this way:
- for $\vec{x}_0$ to the right of ⁄ $\vec{x}(t) \to$ ⁄
- " left " "

ii) the blue separatrix defines the two basins of attraction for $t \to -\infty$:
- $\vec{x}_0$ below ⁄ $x(t) \to$ ⁄
- " " "



> The separatrices are invariant sets for the dynamical system.

Prove that the line $y = x$ is an invariant set for the dynamical system

Define $L = x - y$. The line $L = 0$ is an invariant set for the dynamical system

$$\begin{cases}\dot{x} = 3x - y\\ \dot{y} = 6x - 4y\end{cases}$$

$$\begin{aligned}\text{if } \dot{L} = 0 \quad &\Leftrightarrow \quad \dot{x} - \dot{y} = 0\\ &\Leftrightarrow \quad 3x - y - (6x - 4y) = 0\\ &\Leftrightarrow \quad -3x + 3y = 0\\ &\Leftrightarrow \quad y = x \qquad \blacksquare\end{aligned}$$

L is also called an unstable manifold for the dynamical system

When is a steady state stable?

Solutions of $\dot{\vec{x}} = A\vec{x}$ are of the form:

Case 1 $\vec{x}(t) = \sum_{i=1}^{n} c_i\vec{v}_i e^{\lambda_i t}$ where $\lambda_i \in \mathbb{R}$ are the eigenvalues of A
$\vec{v}_i$ " eigenvectors of A
c_i " constants determined by ICs.
n is the length of $\vec{x}$ + A is $n \times m$

> Solutions $|\vec{x}(t)|$ grow if at least one of the $\lambda_i > 0$ → unstable SS
> decay if all of the $\lambda_i < 0$ → stable SS

Case 2 m pairs of λ_i are complex: $\lambda_i = a_i + ib_i$

$$\therefore \vec{x}(t) = \sum_{i=1}^{m} e^{a_i t}\left(C_{1i}\cos(b_i t) + C_{2i}\sin(b_i t)\right) + \sum_{i=2m+1}^{n} c_i\vec{v}_i e^{\lambda_i t}$$

> Solutions $|\vec{x}(t)|$ grow if at least one of the λ_i or $a_i > 0$ → unstable SS
> decay if all of the λ_i and $a_i < 0$ → stable SS

In 2D

$$\dot{\vec{x}} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\vec{x}$$

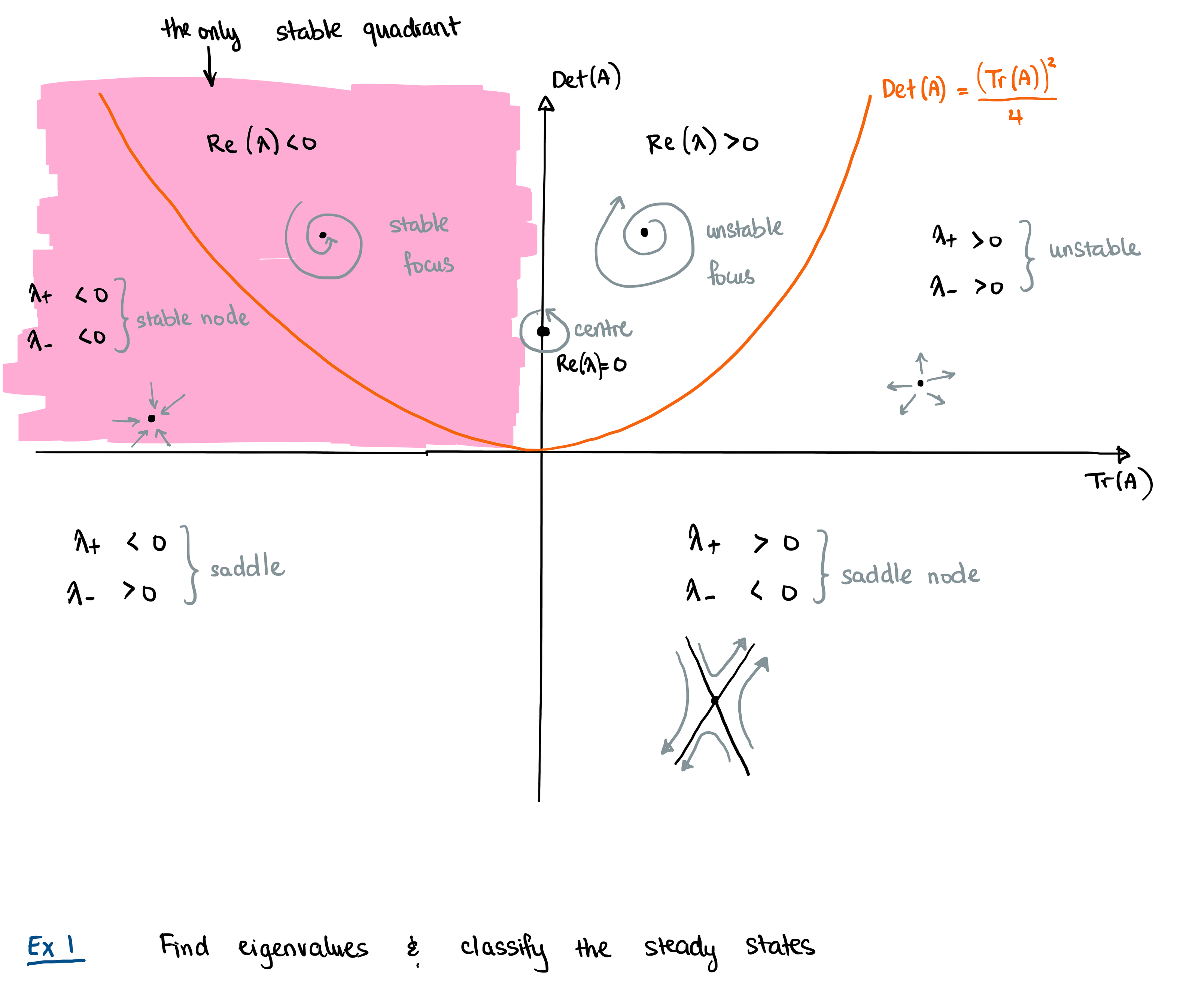
Eigenvalues $\begin{vmatrix}a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{vmatrix} = 0 \quad I$

$$I \Leftrightarrow (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

$$\Leftrightarrow \lambda^2 - \underbrace{(a_{11}+a_{22})}_{\substack{\text{Trace }(A)\\ Tr(A)}} + \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{\substack{\text{Determinant }(A)\\ Det(A)}} = 0$$

$$\therefore \lambda = \frac{Tr(A) \pm \sqrt{(Tr(A))^2 - 4Det(A)}}{2}$$

$$\lambda \in \mathbb{R} \text{ if } (Tr(A))^2 - 4Det(A) \geq 0$$
$$Det(A) \leq \frac{1}{4}(Tr(A))^2$$



Ex 1 Find eigenvalues & classify the steady states

1. $\dot{\vec{x}} = \begin{bmatrix}2 & -4\\ 2 & -2\end{bmatrix}\vec{x}$

$\lambda_1 = 2i$
$\lambda_2 = -2i$

$\therefore (0,0)$ is a centre

2. $\dot{\vec{x}} = \begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}\vec{x}$

$\lambda_1 = 3$
$\lambda_2 = 2$

$\therefore (0,0)$ is an unstable node