Lecture #6

Continuing the examples from last time... Last time we found the eigenvalues & classified the steady states of two linear systems.

In this next example, we will work with a nonlinear system, which will allow us to extend our understanding of the role of separatrices.

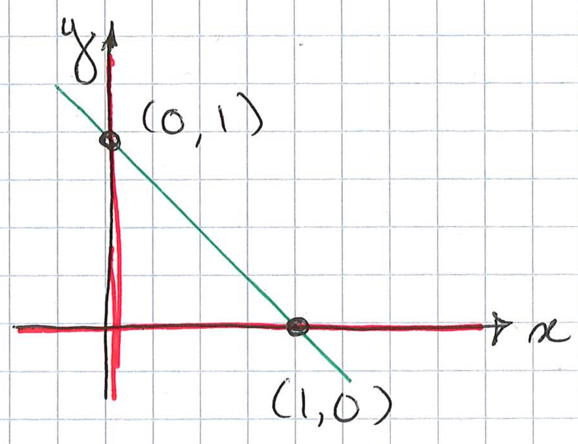
Ex 3

Consider the nonlinear system

$$\begin{cases} \dot{x} = xy & = f(x,y) \\ \dot{y} = 1-x-y & = g(x,y) \end{cases}$$

Nullclines:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \iff \begin{cases} xy = 0 \\ 1-x-y = 0 \end{cases} \iff \begin{cases} x = 0 \text{ or } y = 0 \\ y = 1-x \end{cases}$$



Linearized system:

$$A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} y & x \\ -1 & -1 \end{bmatrix} \; *$$

At $(1,0)$ $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ $\lambda = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{3}\,i}{2}$

$\Rightarrow$ stable focus

At $(0,1)$ $A = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$ $\lambda = \frac{0 \pm \sqrt{0+4}}{2} = \pm 1$

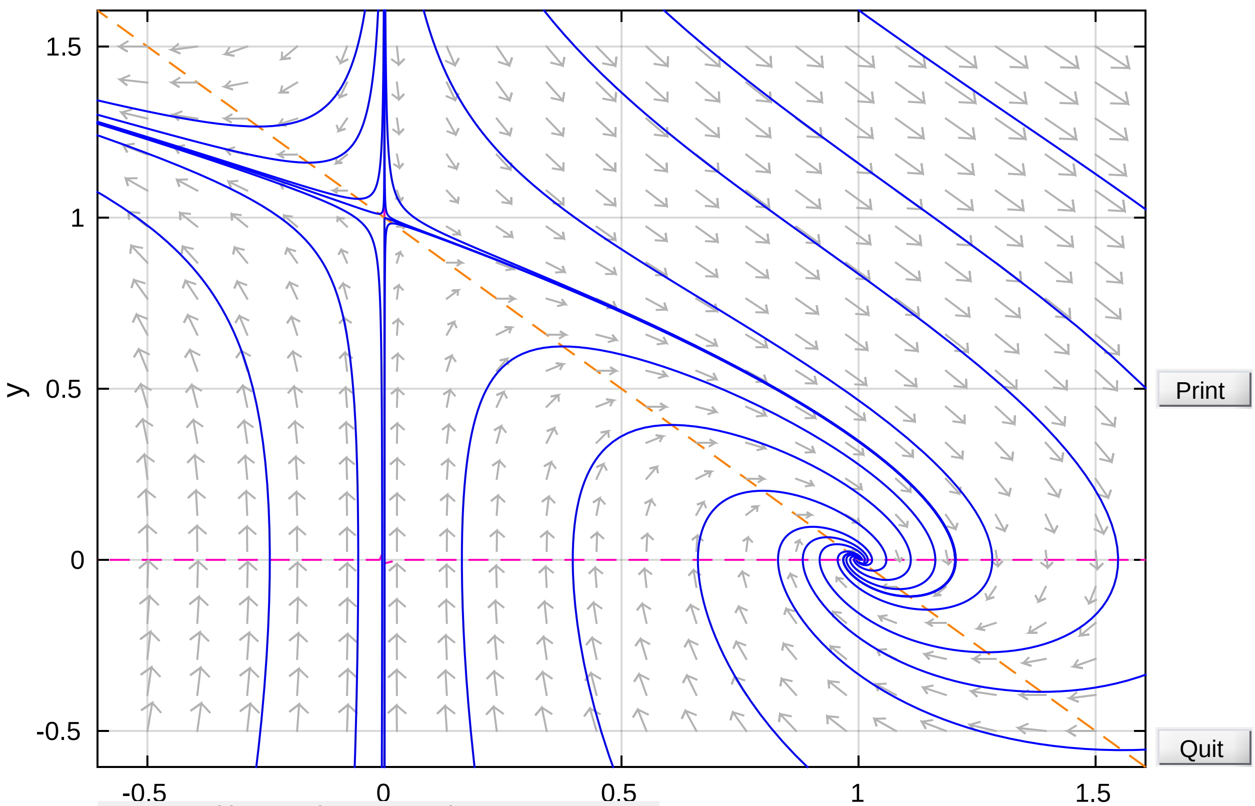
$\Rightarrow$ saddle

Exploring with pplane, we see that there is ~~a~~ a manifold joining the $(1,0)$ + $(0,1)$ steady states (it is an unstable manifold of the $(0,1)$ steady state).

(see the phase plane diagram on the next page)

Trajectories joining steady states are called "heteroclinic orbits".

x ' = x y
y ' = 1 - x - y



Cursor position: (-0.618, 1.86)

The backward orbit from (0.41, 0.19) left the computation window.
Ready.
The forward orbit from (0.66, 0.035) --> a possible eq. pt. near (1, -2.4e-12).
The backward orbit from (0.66, 0.035) left the computation window.
Ready.

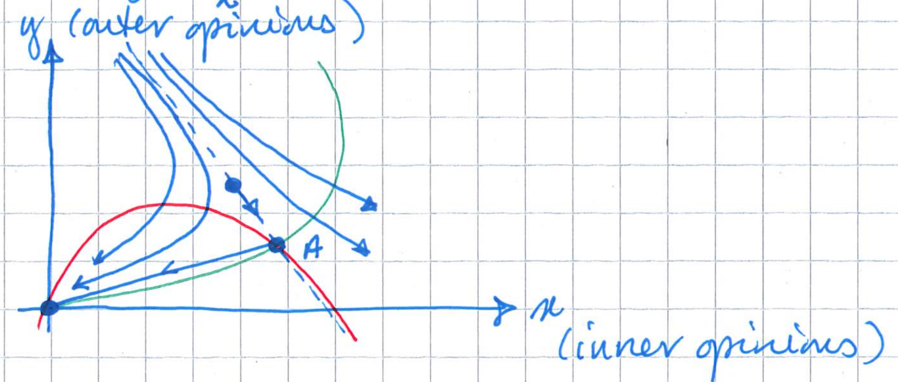
Opinion Dynamics Model

We are now ready to use pplane to explore the Opinion Dynamics model!

Simplified Model (eqn (2) from lecture #2):

$$(23)\cdots \begin{cases} \dot{x} = x(y+x) + y(1-y-p_a x) - x(1-(1-p_a)x) \\ \dot{y} = x(y+p_a x) - y(1-y-p_a x) \end{cases}$$

IC: $x_0 + y_0 = \frac{1}{2}$



What does this mean?

- If $x_0 + y_0 = \frac{1}{2}$, then $x(t)$ & $y(t)$ approach the steady state at A, where $\underbrace{y(t) \ll 1}_{\text{outer opinions are few}}$ & $\underbrace{x(t) \sim \frac{1}{2}}_{\text{inner opinions dominate}}$.

  centering

- $\because$ A is a saddle the flow will eventually zoom off either toward $(0,0)$, or toward $(1,0)$.

Once the flow leaves the line $x+y=\frac{1}{2}$, the simplified model is no longer relevant, so we need a different one.

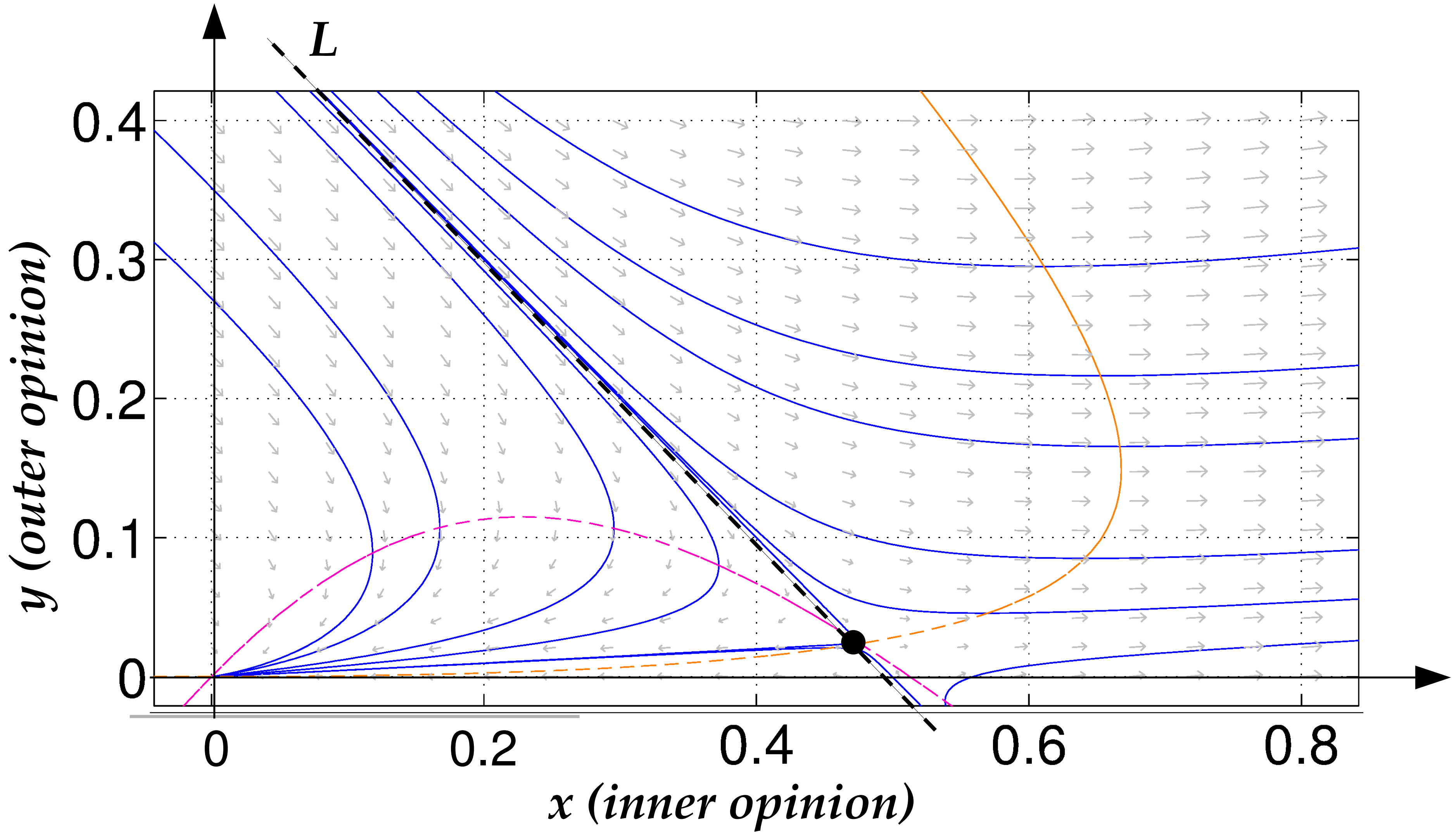
Opinion Dynamics Simplification #2:

let $\begin{cases} L_1 = L_2 = 0 \\ R_1 = x_r,\ R_2 = y_r \end{cases}$

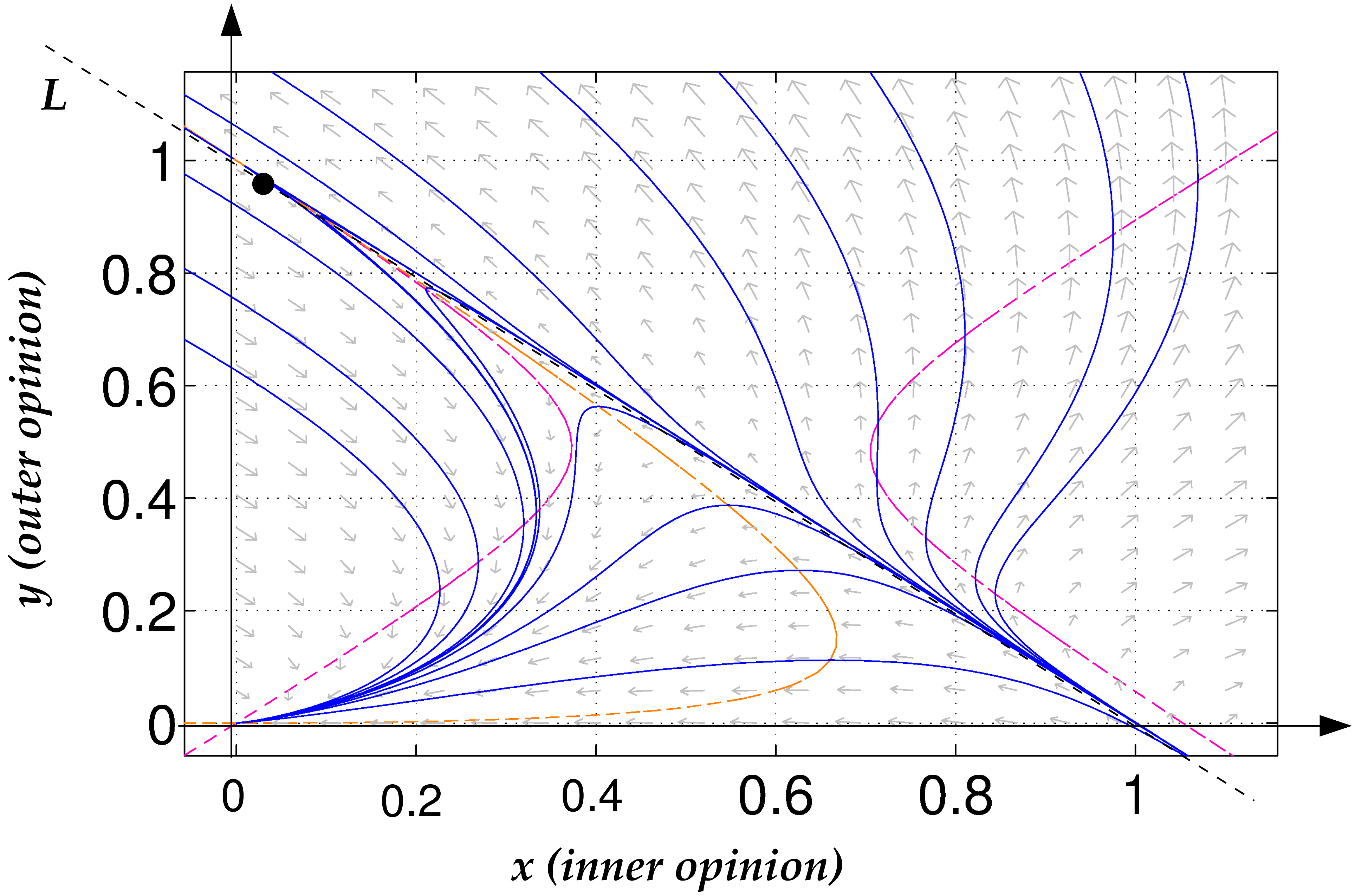
then we have

$$(24) \ldots \quad \begin{cases} \dot{x}_r = y_r(1-y_r-p_a x_r) - x_r(1-(1-p_a)x_r) \\ \dot{y}_r = x_r(y_r+p_a x_r) - y_r(1-y_r-p_a x_r) \end{cases}$$

(See the phase plane diagrams on the next two pages. The first phase plane ~~doesn't~~ corresponds to the $L_1=R_1$, $L_2=R_2$ system (23), and the second phase plane corresponds to the $L_1=L_2=0$, $R_1=x_r$, $R_2=y_r$ system (24).)



L
0.4
0.3
0.2
0.1
0
y (outer opinion)
0
0.2
0.4
0.6
0.8
x (inner opinion)



L
1
0.8
0.6
0.4
0.2
0
y (outer opinion)
0
0.2
0.4
0.6
0.8
1
x (inner opinion)

We discussed how the two phase planes (for (23) + (24)) explain the results seen in Figure 4 of Baumgaertner et al (2018). Consider first the solid lines in that figure.

- The phase plane for (23) shows that for initial conditions close to $x = y = \frac{1}{4}$ (i.e. $L_1 = L_2 = R_1 = R_2 = \frac{1}{4}$), the system follows the $x + y = \frac{1}{2}$ separatrix toward the steady state near ($x = \frac{1}{2}$, $y = 0$). This explains the initial centering seen in Figure 4.

- Since the $(\frac{1}{2}, 0)$ steady state is a saddle, we see that eventually the system (23) will shoot off toward $(0, 0)$ or (x large, y small). This explains the drop towards zero of the R_1 + R_2 opinions in Figure 4, + the rapid increase of the L_1 opinions.

- Now the phase plane for (24) becomes relevant. The phase plane for (23) tells us that the appropriate starting point on the second phase plane is one where x is large + y is small. Since we have assumed that opinions on one side are zero, we must have $x + y = 1$. So we start on the line $x + y = 1$, near the lower left.

- The phase plane for (24) tells us that the initial point on $x + y = 1$ will move toward the steady state at the top left, where $x \to 0$ + $y \to 1$. This explains the consensus behavior in Figure 4.

Finally if we consider all of the curves in Figure 4 we notice that centering is maintained longer under some scenarios than others. This behaviour can be obtained in two ways

1) decrease amplification
   (echo-chambers lead to rapid swings of opinion to one side or another!)
2) start closer to the separatrix for the steady state near $(\frac{1}{2}, 0)$ in the phase plane for (23), by making the initial condition closer to $L_1 = L_2 = R_1 = R_2 = \frac{1}{4}$ (give a smaller advantage to one side), so that the system is closer to the heteroclinic trajectory.

Discussion

- Simple models can be strong tools for explaining more complex models.
- Even simple models can help explain systems that look very complex, such as the evolution of opinions in a population.
- This model shows that the rapid evolution of opinions toward one extreme or another happens faster if amplification is increased, or mixing is increased (see the paper).