Lecture 6

Tuesday, September 24, 2019 3:29 PM

Continuing from last time ....

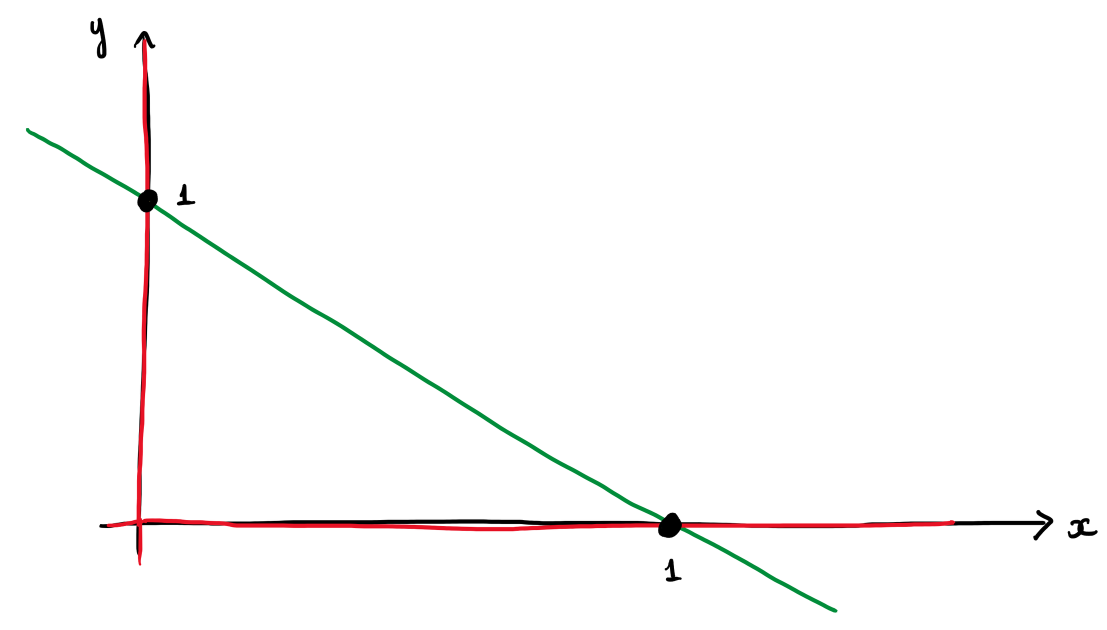
In this next example, we will work in a non-linear system & extend our understanding of separatrices.

Ex 3

$$\begin{cases} \dot{x} = xy & = f(x,y) \\ \dot{y} = 1 - x - y & = g(x,y) \end{cases}$$

Nullclines

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \iff \begin{cases} x = 0 \quad \text{or} \quad y = 0 \\ y = 1 - x \end{cases}$$



Linearized System

$$\dot{x} = A\vec{x} \quad , \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

A = "community matrix"
= "jacobian"

$$= \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} y & x \\ -1 & -1 \end{bmatrix}$$

At $(1,0)$:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad \lambda = \frac{1 \pm \sqrt{3}\, i}{2} \quad \Rightarrow \text{ stable focus}$$

At $(0,1)$

$$A = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \qquad \lambda = \frac{0 \pm \sqrt{0+4}}{2} = \pm 1 \quad \Rightarrow \text{ saddle}$$

Eigenvectors:

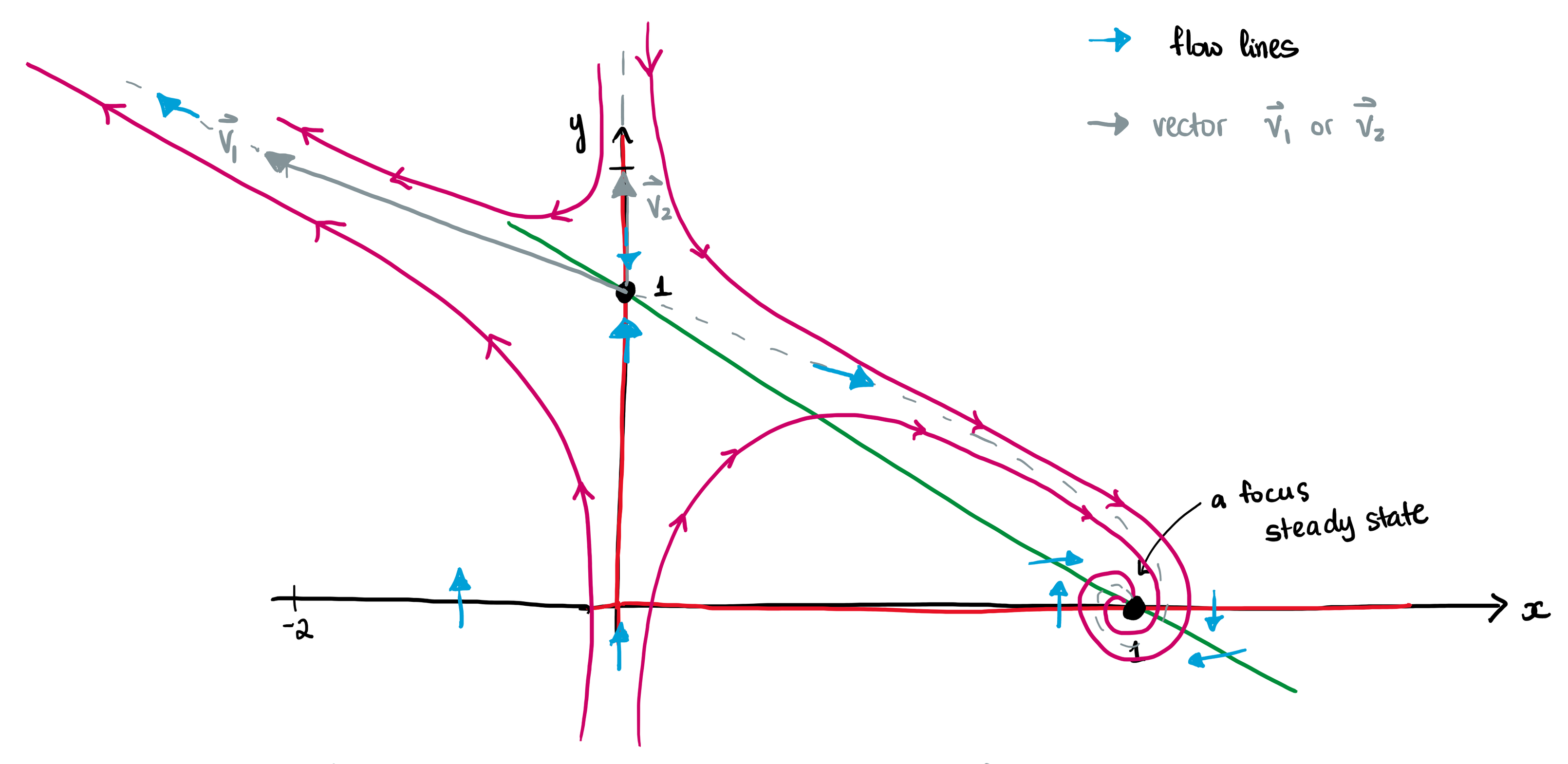
$$\lambda_+ = 1 \qquad A\vec{v} = \lambda\vec{v} \iff \begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-v_{11} - 2v_{12} = 0$$

Choose: $v_{11} = 1 \Rightarrow v_{12} = -2$ $\qquad \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\lambda_- = -1 \qquad [A - \lambda I]\vec{v}_2 = \vec{0} \iff \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

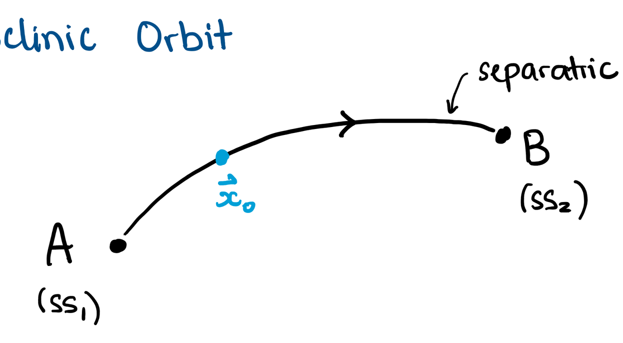
Choose: $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



The separatrix $\vec{v}_1$ spirals into the steady state $(1,0)$

In a non-linear system, separatrices may join steady states. Such a trajectory is called a "heteroclinic orbit." Such trajectries no longer define basins of attraction.

Heteroclinic Orbit



Given an initial condition $\vec{x}_0$ that lies on the heteroclinic trajectory,

$$\lim_{t \to +\infty} \vec{x}(t) = A$$

$$\lim_{t \to -\infty} \vec{x}(t) = B$$

Opinion Dynamics Model

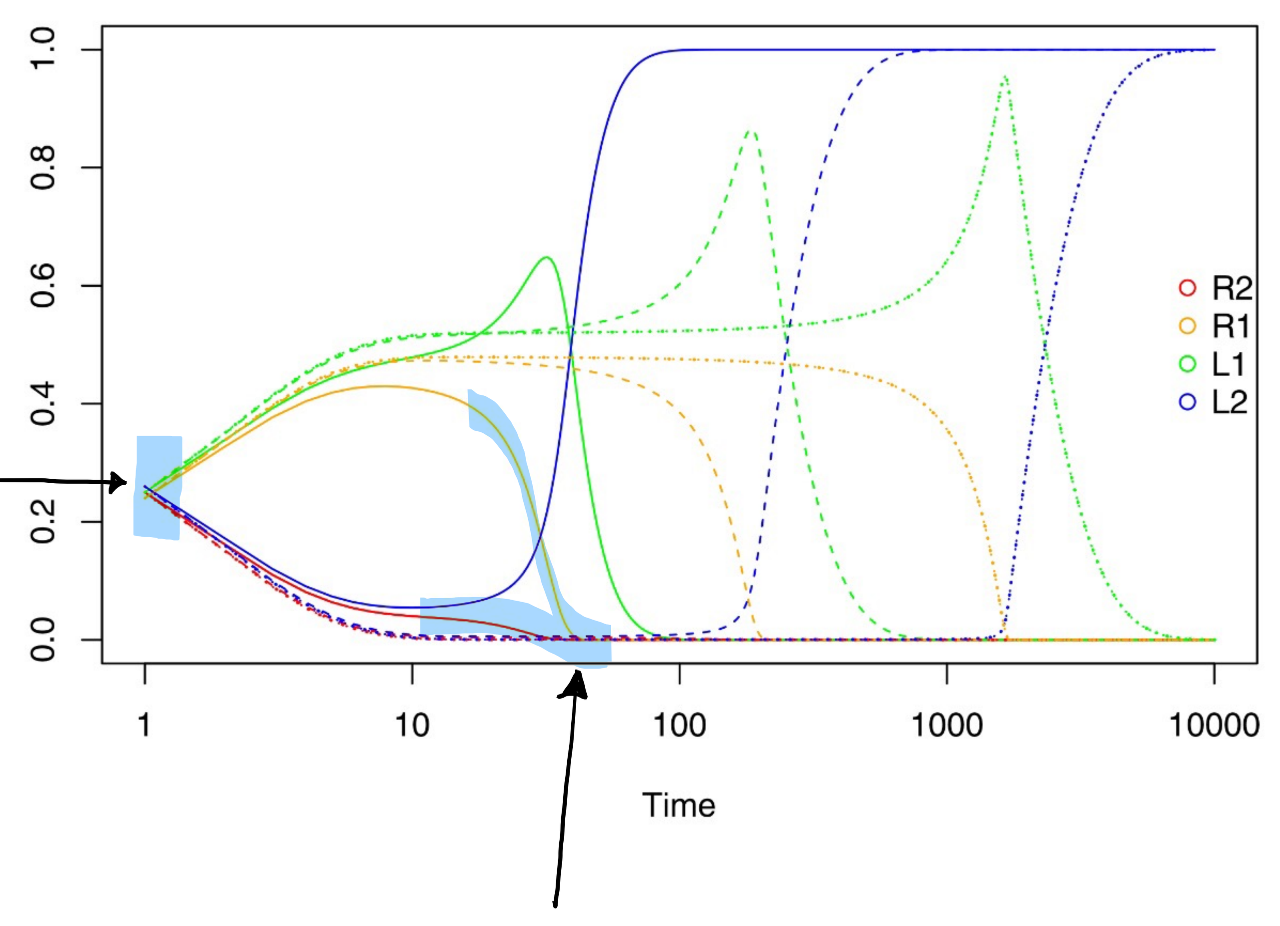
original model : 4-dimensional

Solutions

initially, simplification #1

$L_1 = R_1$; $L_2 = R_2$

$\Rightarrow$ 2-dimensional model



simplification #2 :

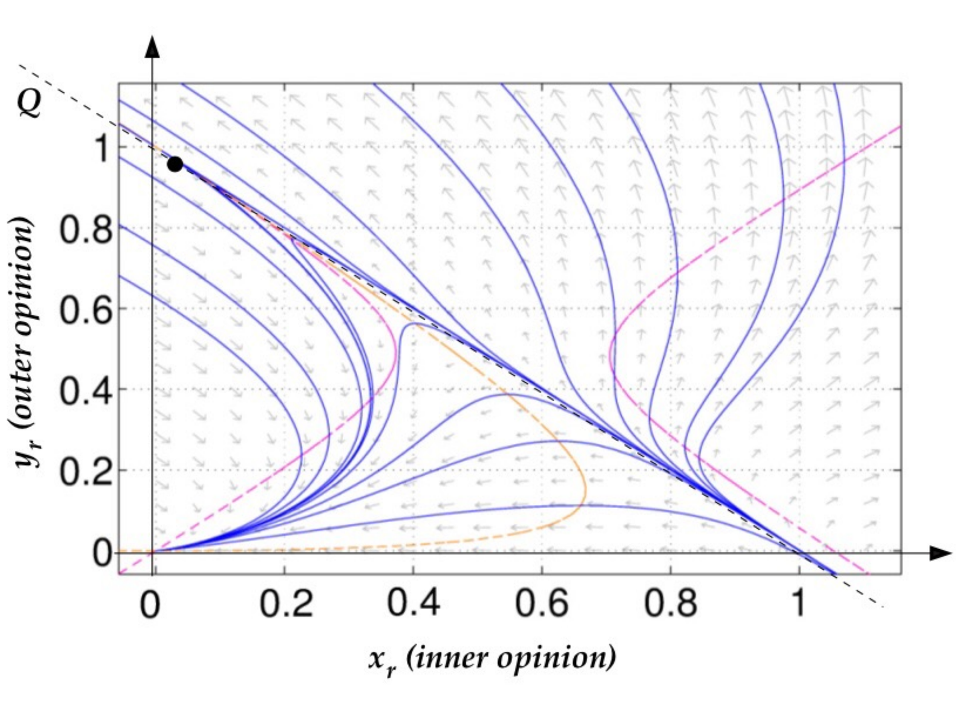
$L_1 = L_2 = 0$

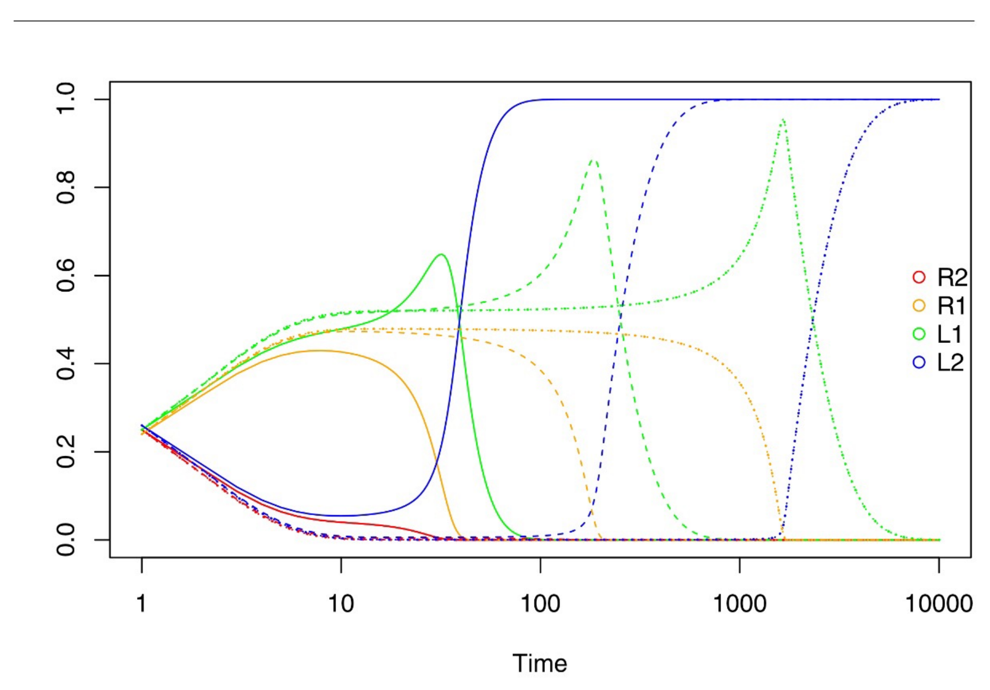
$R_1 = x_r$

$R_2 = y_r$

We obtain:

$$\begin{cases} \dot{x}_r = y_r(1 - y_r - p_a x_r) - x_r(1 - (1 - p_a)x_r) \\ \dot{y}_r = x_r(y_r + p_a x_r) - y_r(1 - y_r - p_a x_r) \end{cases}$$





- If we start on the dashed-black line (separatrix), we will go towards the steady state at $(0,1)$ $\Rightarrow$ consensus