Lecture #7

Wrapping up from last time:

Opinion Dynamics

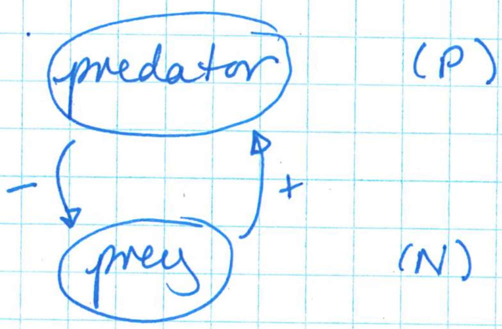
- ABM = very little mixing ⇒ polarization (see the other research paper)
- ODE = maximum mixing ⇒ swing to consensus

Concl:
- mixing is important for consensus
- internet is not promoting mixing
  - ▹ targeted ads
  - ▹ recommendations for next youtube watch
  - ▹ no supervision

Global behaviour from local information

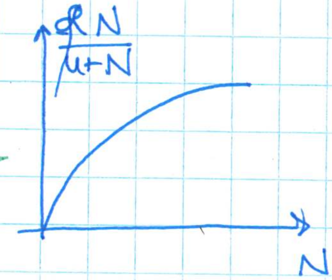
- systems of nonlinear ODEs may have multiple steady states
- close to the steady states, behaviour is approximated by the linearized equations
  - ⇝ holds true for $n \times n$ systems!
- special attribute of 2×2 systems
  - ⇝ local behaviour at steady states can be used to reconstruct the global behaviour b/c
    - ▹ solution curves can only intersect at steady states (uniqueness)
    - ▹ if a solution curve is a closed loop, it must encircle at least one steady state that cannot be a saddle

Predator - Prey Systems



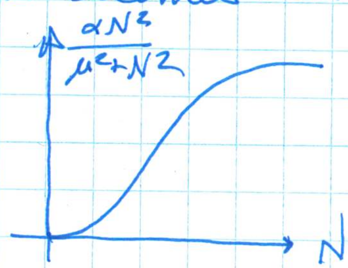
Specialist predator (bananivore!) relies on one food source for survival:

$$\begin{cases} \dfrac{dN}{dt} = f(N) - \dfrac{\alpha N}{\mu + N} P \\[2ex] \dfrac{dP}{dt} = \chi \dfrac{\alpha N}{\mu + N} P - \delta P \end{cases}$$



Generalist predator (switches to other fruits when bananas become scarce or expensive) can survive on a variety of food sources & loses interest in the focal one when it becomes scarce:

$$\begin{cases} \dfrac{dN}{dt} = f(N) - \dfrac{\alpha N^2}{\mu^2 + N^2} P \\[2ex] \dfrac{dP}{dt} = \chi \dfrac{\alpha N^2}{\mu^2 + N^2} P - \delta P \end{cases}$$



What if the predator is a specialist in the winter & generalist in the summer? This is the case for boreal great horned owls. Lengthening summers could have an important effect!

In order to understand the seasonal system, we begin by studying a specialist predator model:

$$\text{prey:} \quad \frac{dN}{dT} = rN\left(1-\frac{N}{K}\right) - \frac{cNP}{a+N}$$

$$\text{pred:} \quad \frac{dP}{dT} = \frac{bNP}{a+N} - mP \quad \cdots \quad (1)$$

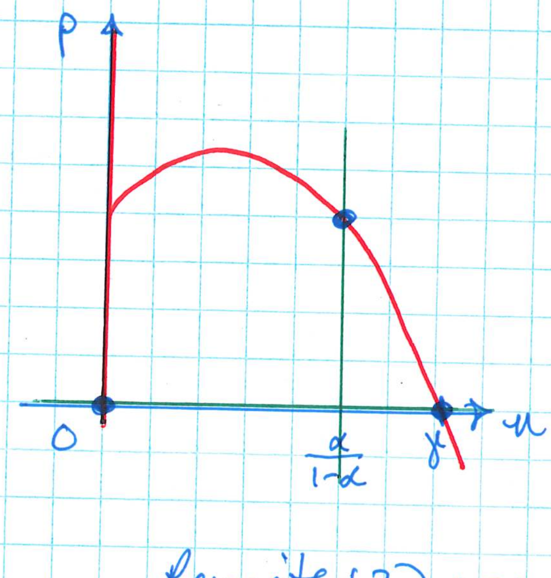
Let

$$n = \frac{N}{a}, \quad p = \frac{c}{ra}P, \quad t = rT$$

$$\alpha = \frac{m}{b}, \quad \beta = \frac{b}{r}, \quad \gamma = \frac{K}{a}$$

Then (1) becomes

$$\begin{cases} \dot{n} = n\left(1-\dfrac{n}{\gamma}\right) - \dfrac{np}{1+n} \\ \dot{p} = \beta\left(\dfrac{n}{1+n} - \alpha\right)p \end{cases} \quad \cdots \quad (2)$$



$(n^*, p^*) = (0,0)\ ;\ (\gamma, 0)\ ;$

$\left(\frac{\alpha}{1-\alpha}, g\left(\frac{\alpha}{1-\alpha}\right)\right)$

where $g(\cdot)$ is defined on the next page

Rewrite (2) as

$$\begin{cases} \dot{n} = f(n)\left[g(n) - p\right] \\ \dot{p} = \beta\left[f(n) - \alpha\right]p \end{cases} \quad \cdots \quad (3)$$

where

$$f(u) = \frac{u}{1+u}, \quad g(u) = \underbrace{(1+u)\left(1-\frac{u}{\gamma}\right)}_{\text{this is the equation of the quadratic prey nullcline}} \quad \cdot \quad \cdot \qquad (4)$$

Then

$$A = \begin{bmatrix} f(u)g'(u) + f'(u)g(u) - pf'(u) & -f(u) \\ \beta f'(u)p & \beta[f(u)-\alpha] \end{bmatrix}$$

At $\underline{(0,0)}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha\beta \end{bmatrix} \Rightarrow \text{saddle}$$

At $\underline{(\gamma,0)}$

$$A = \begin{bmatrix} -1 & -f(\gamma) \\ 0 & \beta(f(\gamma)-\alpha) \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= \beta\left(\frac{\gamma}{1+\gamma} - \alpha\right) \end{aligned}$$

$\Rightarrow$ stable node if $\dfrac{\gamma}{1+\gamma} < \alpha \Leftrightarrow \gamma < \dfrac{\alpha}{1-\alpha}$ $+ (1-\alpha > 0)$

$\Rightarrow$ saddle if $\dfrac{\gamma}{1+\gamma} > \alpha \Leftrightarrow \gamma > \dfrac{\alpha}{1-\alpha}$ $(1-\alpha > 0)$

At $\underline{\left(u^*, (1+u^*)\left(1-\frac{u^*}{\gamma}\right)\right), \ u^* = \frac{\alpha}{1-\alpha}}$

$$A = \begin{bmatrix} \alpha g'(u^*) & -\alpha \\ \beta f'(u^*)g(u^*) & 0 \end{bmatrix}$$

Characteristic equation:

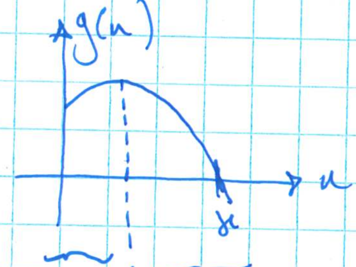
$$\lambda^2 - \alpha g'(u^*)\lambda + \alpha\beta f'(u^*) g(u^*) = 0 \quad \ldots\ldots \quad (5)$$

For this steady state to be stable, we require $Re(\lambda) < 0,\ \forall \lambda$? $\Rightarrow Tr(A) < 0$ + $Det(A) > 0$.

$$\begin{cases} Tr(A) = \alpha g'(u^*) \\ Det(A) = \alpha\beta f'(u^*) g(u^*) \quad > 0 \text{ for } u^* > 0 \end{cases} \quad \ldots (6)$$

Consider

$g'(u^*)$:

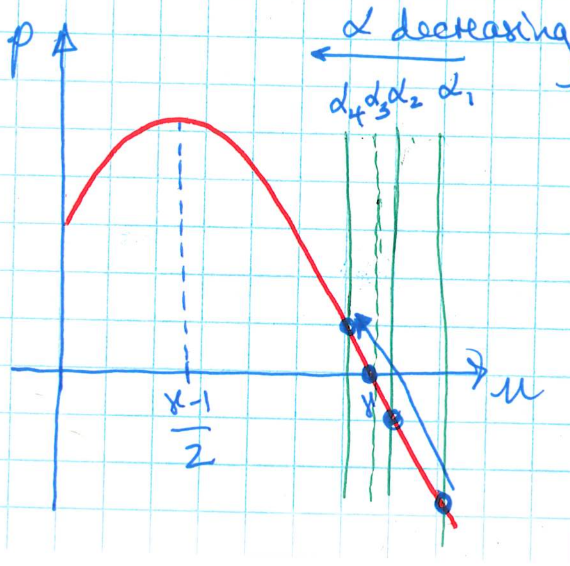


$Tr(A) < 0$, steady state stable $\Leftarrow$ here $g' < 0$

here $g' > 0$ $\Rightarrow$ $Tr(A) > 0$, steady state unstable node

here, $g' = 0$
(pure imaginary eigenvalues)
$\Rightarrow$ centre
$\left(u = \frac{\gamma - 1}{2}\right)$

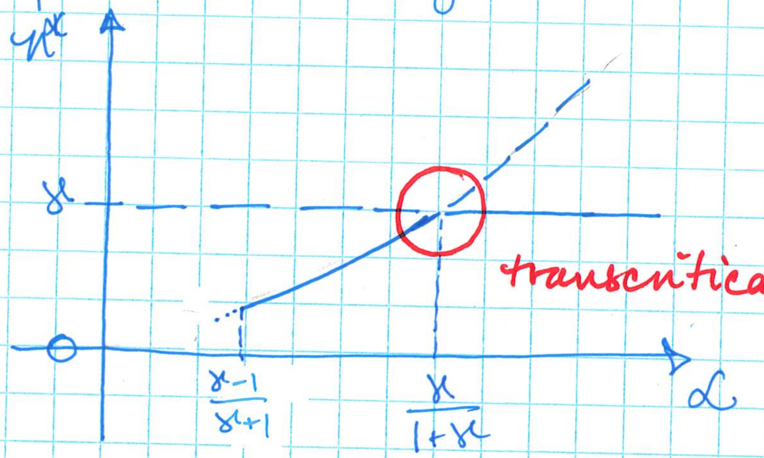
Let's put together the $(\gamma, 0)$ & $\left(\frac{\alpha}{1-\alpha}, g\left(\frac{\alpha}{1-\alpha}\right)\right)$ information:



α	stability of $(\gamma, 0)$	stability of $\left(\frac{\alpha}{1-\alpha}, g\left(\frac{\alpha}{1-\alpha}\right)\right)$
α_1	stable node	saddle
α_2	"	"
α_3	?	?
α_4	saddle	stable (node or focus)

So there is an exchange of stability when the $\left(\frac{\alpha}{1-\alpha}, g\left(\frac{\alpha}{1-\alpha}\right)\right)$ steady state passes through the $(\gamma, 0)$ steady state.

Bifurcation diagram:



$x^* = 0$ or γ

or $\frac{\alpha}{1-\alpha}$

– – – = unstable steady state

——— = stable " "

What happens at $\alpha = \frac{\gamma-1}{\gamma+1}$ (when the predator nullcline is at the peak of $g(x)$)? Using pplane, we saw that for $\alpha < \frac{\gamma-1}{\gamma+1}$, the steady state solution is a periodic orbit called a <u>limit cycle</u>.