Lecture 7

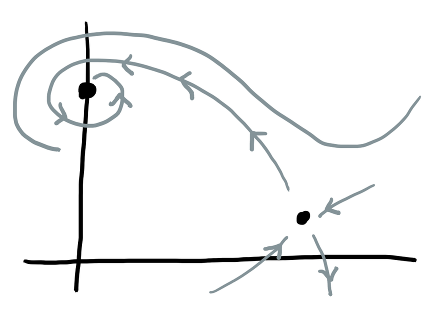
Thursday, September 26, 2019 3:36 PM

Opinion Dynamics

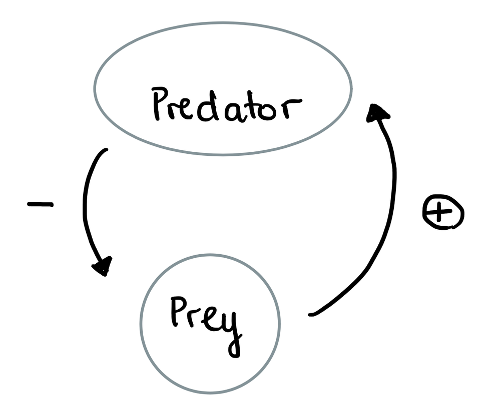
- mixing is important for consensus
- little mixing $\Rightarrow$ polarization (esp. if amplification is present)

Global Behaviour from Local Solutions

- system of non-linear ODE can have multiple steady states
- close to the steady states, behaviour is approximated by the linearized system
  $\Rightarrow$ general (good for $n \times n$)
- special attributes of 2×2 systems
  $\rightsquigarrow$ global behaviour can be inferred from local
  - solution curves can only intersect at steady state
  - if a solution curve is a closed loop, it must contain at least 1 steady state that is not a saddle (theory for later)



Predator - Prey Systems



Specialist Predator

$$\left.\begin{aligned} \text{prey} &= \frac{dN}{dT} = rN\left(1-\frac{N}{K}\right) - \frac{cNP}{\alpha+N} \\ \text{pred} &= \frac{dP}{dt} = \frac{bNP}{a+N} - mP \end{aligned}\right\} \quad \dots \quad (1)$$

Let $\mu = \frac{N}{a}$, $p = \frac{c}{ra}P$, $t = rT$

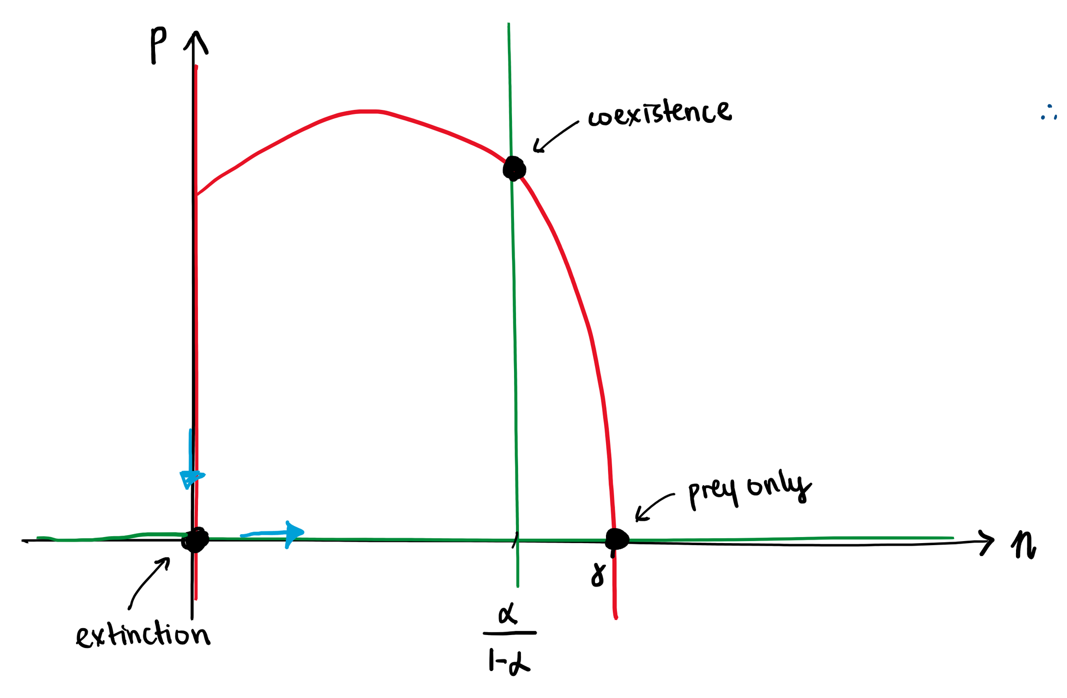
$\alpha = \frac{m}{b}$, $\beta = \frac{b}{r}$; $\gamma = \frac{K}{a}$ > 0

Then (1) becomes

$$\dot{n} = n\left(1-\frac{n}{\gamma}\right) - \frac{np}{1+n}$$

$$\dot{p} = \beta\left(\frac{n}{1+n} - \alpha\right)p$$

Nullclines



$\therefore$ 3 steady states.

$(0,0)$

$(\gamma, 0)$

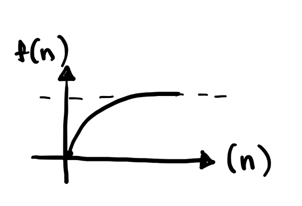
$\left(\frac{\alpha}{1-\alpha}, g\left(\frac{\alpha}{1-\alpha}\right)\right)$

Linearization

$$\left.\begin{aligned} \dot{n} &= f(n)\left(g(n) - p\right) \\ \dot{p} &= \beta\left(f(n) - \alpha\right)p \end{aligned}\right\} \quad \dots\dots \quad (3)$$

where

$$\left.\begin{aligned} f(n) &= \frac{n}{1+n} \\ g(n) &= (1+n)\left(1-\frac{n}{\gamma}\right) \end{aligned}\right\} \quad \dots\dots \quad (4)$$



$\Rightarrow$ f' is positive

Then $A = J$

$$= \begin{bmatrix} f(n)g'(n) + f'(n)g(n) - pf'(n) & -f(n) \\ \beta f'(n)p & \beta(f(n)-\alpha) \end{bmatrix}$$

Plug in Steady States.

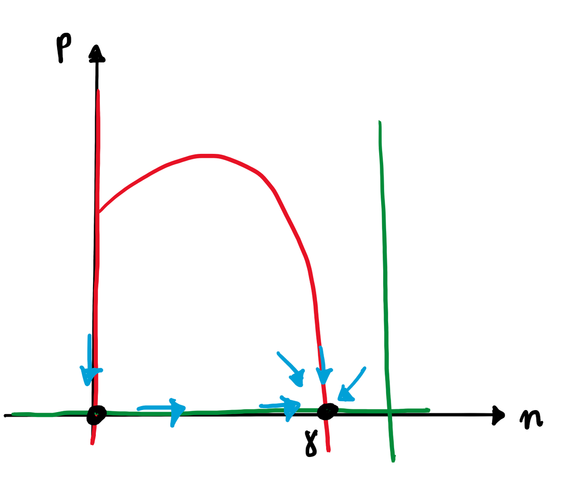
at (0,0) $A = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha\beta \end{bmatrix}$ $\Rightarrow$ saddle point

$\therefore$ extinction is never an option

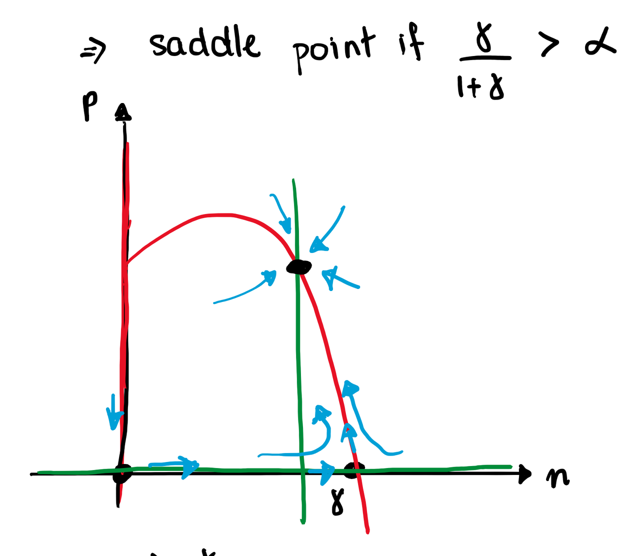
at $(\gamma, 0)$ $A = \begin{bmatrix} -1 & -f(\gamma) \\ 0 & \beta(f(\gamma)-\alpha) \end{bmatrix}$ $\Rightarrow$ stable node if $\frac{\gamma}{1+\gamma} < \alpha$

↳ prey nullcline lies to the right of $(\gamma, 0)$

$\Rightarrow$ saddle point if $\frac{\gamma}{1+\gamma} > \alpha$



if $\frac{\gamma}{1+\gamma} < \alpha$



if $\frac{\gamma}{1+\gamma} > \alpha$

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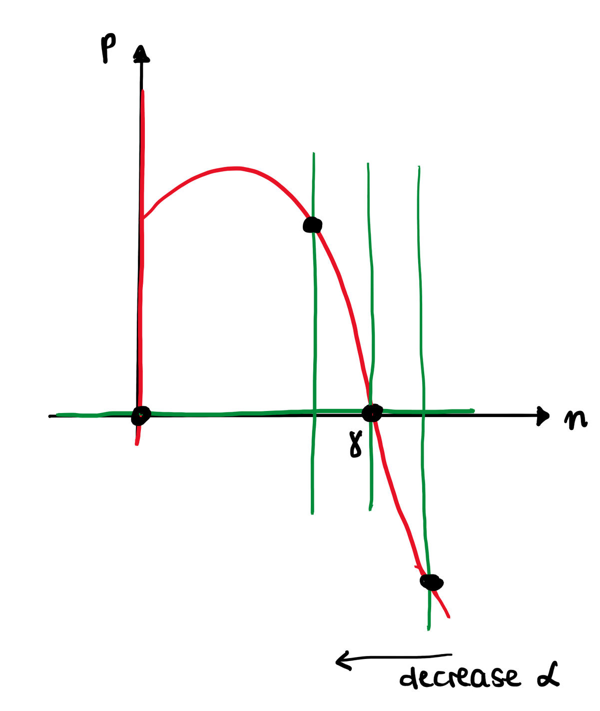


Bifurcation

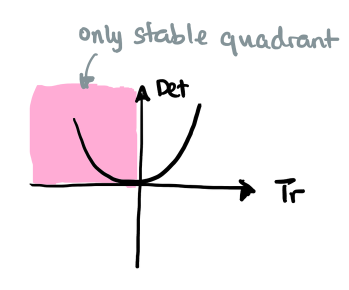
change in stability of a steady state

In this case, one steady state goes through another one and there is an exchange of stability (explained below).

At $\left(\frac{\alpha}{1-\alpha}, g\left(\frac{\alpha}{1-\alpha}\right)\right)$

$$A = \begin{bmatrix} \alpha g'\left(\frac{\alpha}{1-\alpha}\right) & -\alpha \\ \beta f'\left(\frac{\alpha}{1-\alpha}\right) g\left(\frac{\alpha}{1-\alpha}\right) & 0 \end{bmatrix}$$

only stable quadrant



Character Eq.

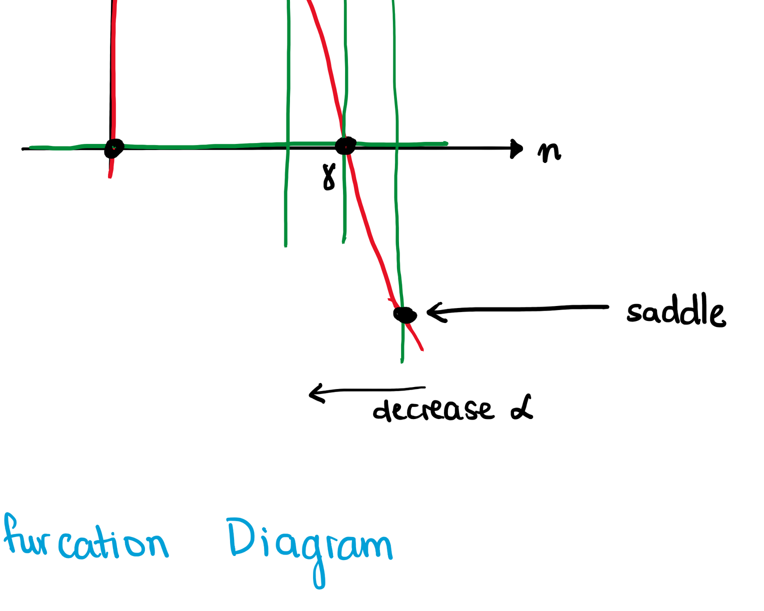
$$\lambda^2 - \underbrace{\alpha g'\left(\frac{\alpha}{1-\alpha}\right)}_{\mathrm{Tr}(A)}\lambda + \underbrace{\alpha\beta f'\left(\frac{\alpha}{1-\alpha}\right) g\left(\frac{\alpha}{1-\alpha}\right)}_{\det(A)} = 0$$

- $\mathrm{Det}(A) > 0$ for $0 < n < \gamma$
  ↳ as soon as $\mathrm{Det}(A) < 0$ $\Rightarrow$ saddle points.
- $g'\left(\frac{\alpha}{1-\alpha}\right)$ is positive for small n, negative " large n } sign depends on g'

↳ when $g'\left(\frac{\gamma}{1-\alpha}\right) < 0$ $\Rightarrow$ $\mathrm{Tr}(A) < 0$ $\Rightarrow$ stable node / focus.

↳ " > 0 $\Rightarrow$ $\mathrm{Tr}(A) > 0$ $\Rightarrow$ unstable node / focus

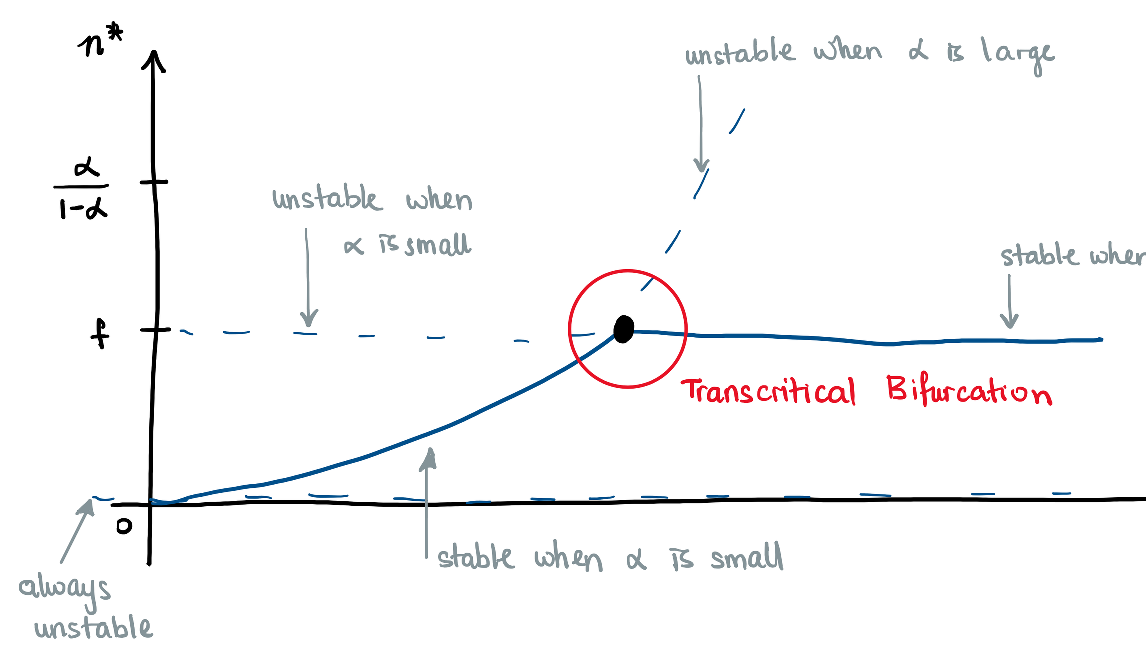
↳ " $= 0$ $\Rightarrow$ $\mathrm{Tr}(A) = 0$ $\Rightarrow$ centre.



	α large	α smaller
$(\gamma, 0)$	stable node	saddle point
(coexistence)	saddle	stable

exchange of stability

Bifurcation Diagram



- - - unstable

— stable